$$\int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+\tfrac{1}{4}\left(e^{2x}-2+e^{-2x}\right)}\,dx = \int_0^3 \tfrac{1}{2}\sqrt{4+e^{2x}-2+e^{-2x}}\,dx$$

$$= \int_0^3 \tfrac{1}{2}\sqrt{e^{2x}+2+e^{-2x}}\,dx = \int_0^3 \tfrac{1}{2}\sqrt{\left(e^{x}+e^{-x}\right)^2}\,dx = \int_0^3 \tfrac{1}{2}\left(e^{x}+e^{-x}\right)dx$$

$$= \tfrac{1}{2}\left[e^{x}-e^{-x}\right]_0^3 = \tfrac{1}{2}\left(e^{3}-e^{-3}-1+1\right) = \tfrac{1}{2}\left(e^{3}-e^{-3}\right)$$

Question 2

Find the length of the curve $r = e^{\theta}$ from $\theta = 0$ to $\theta = \pi$.

Solution

$$\int ds = \int \sqrt{dr^2 + r^2 d\theta^2} = \int \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta. \qquad r = e^{\theta} \qquad \frac{dr}{d\theta} = e^{\theta}$$

$$\int_0^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \int_0^{\pi} \sqrt{e^{2\theta}+e^{2\theta}}\,d\theta = \int_0^{\pi} \sqrt{2e^{2\theta}}\,d\theta = \int_0^{\pi} \sqrt{2}\,e^{\theta}\,d\theta$$

$$= \left[\sqrt{2}\,e^{\theta}\right]_0^{\pi} = \sqrt{2}\left(e^{\pi}-1\right)$$

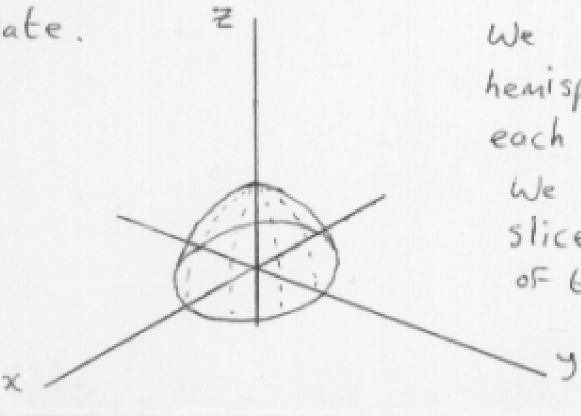
Question 3

Find the centre of gravity of a solid hemisphere of constant density and radius 1, bounded above by $x^2+y^2+z^2=1$ and below by the x-y plane.

Solution

The equation $x^2+y^2+z^2=1$ tells us that we are dealing with a hemisphere of radius 1 that is centred at the origin.

Because the mass distribution is symmetric about the origin in both the x and y directions, the x and y components of the location of the centre of gravity, must both be equal to zero. $\bar{x}=0$, $\bar{y}=0$. This leaves us only the z-coordinate to calculate.



We shall proceed by slicing the hemisphere up into circular slices, each parallel to the x-y plane. We shall calculate the mass of each slice, multiply the mass by the z-coordinate of the slice and then sum over all the slices. This will give us the numerator in the expression for $\bar{z}$.