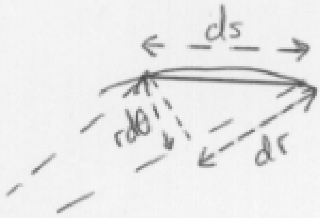
We can divide the curve up into section, in a similar fashion to when we dealt with x-y coordinates. Each section we approximate with a straight line. We calculate the length of each line and sum over all the sections.



$\sum_i ds_i$

In the limit where ds, $d\theta$ and dr become infinitely small, this becomes $\int ds$

$= \int \sqrt{dr^2 + r^2 d\theta^2}$. Factoring $d\theta$ out of the square root gives $\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$. We can calculate $\left(\frac{dr}{d\theta}\right)^2$ because the question must give us an expression for $r(\theta)$, so as long as we are told the limits on θ, we have an integral in θ that we can evaluate.

Question 1

Find the length of the curves

a) $x = \frac{1}{3}(y^2+2)^{3/2}$ from $y=0$ to $y=1$

b) $y = \frac{1}{2}(e^x + e^{-x})$ from $x=0$ to $x=3$.

Solution

a)
$$\int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \qquad x = \frac{1}{3}(y^2+2)^{3/2}$$

$$\frac{dx}{dy} = \frac{1}{3}\,\frac{3}{2}(y^2+2)^{1/2}\, 2y = y(y^2+2)^{1/2}$$

$$\int_0^1 \sqrt{1 + \left(y(y^2+2)^{1/2}\right)^2}\, dy = \int_0^1 \sqrt{1 + y^2(y^2+2)}\, dy = \int_0^1 \sqrt{1 + y^4 + 2y^2}\, dy$$

$$= \int_0^1 \sqrt{(y^2+1)^2}\, dy = \int_0^1 (y^2+1)\, dy = \left[\frac{1}{3}y^3 + y\right]_0^1 = \frac{1}{3} + 1 = \frac{4}{3}$$

b)
$$\int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad y = \frac{1}{2}(e^x + e^{-x}) \quad \frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^x - e^{-x})(e^x - e^{-x}) = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$