Tutorial 14

Linear Independence

A 'Linear Combination' of vectors is an expression that contains a set of vectors, all added together, subtracted from each other or multiplied by scalars. The squares, cubes and other, higher order terms of vectors are not included. Neither are products etween vectors.

For example, if $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ are 3 vectors,

$$\underline{v}_1 + 2\underline{v}_2 - 5\underline{v}_3$$ is a linear combination of them. However,

$$\underline{v}_1 + \underline{v}_1 \cdot \underline{v}_2 + \underline{v}_1 \cdot \underline{v}_3 + 5\underline{v}_3^3$$ is not.

A set of vectors is called linearly independent if we cannot create one of the vectors as a linear combination of the others. So, for our 3 vectors $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ they are called linearly independent if we cannot find two scalars, α and β so that $\underline{v}_1 = \alpha \underline{v}_2 + \beta \underline{v}_3$. If we can find values for α and β that satisfy this equation, the set $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ are called linearly dependent.

Another way to phrase this is to look at the linear combination $\alpha \underline{v}_1 + \beta \underline{v}_2 + \gamma \underline{v}_3 = 0$. If the only solution to this equation is one where $\alpha = 0, \beta = 0, \gamma = 0$, then $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ must be linearly independent. However, if a solution exist in which either $\alpha \neq 0$, $\beta \neq 0$ or $\gamma = 0$, then $\underline{v}_1$, $\underline{v}_2$ and $\underline{v}_3$ must be linearly dependent.

As well as vectors, the question of linear independence can be asked of matrices and functions.

3 matrices, M_1, M_2 and M_3, are linearly independent if $\alpha M_1 + \beta M_2 + \gamma M_3 = 0$ only has the solution $\alpha = 0, \beta = 0, \gamma = 0$. Otherwise, M_1, M_2 and M_3 are linearly dependent.

3 functions, $f_1(x)$, $f_2(x)$ and $f_3(x)$, are linearly independent if $\alpha f_1(x) + \beta f_2(x) + \gamma f_3(x) = 0$ has only the solution $\alpha = 0, \beta = 0, \gamma = 0$. Otherwise, they are linearly dependent.

There are some cases in which a set of vectors cannot be linearly independent. If there are more vectors, in the set, than dimensions in the space, the set of vectors cannot be linearly independent.