Question 4

If $\underline{u}$ and $\underline{v}$ are vectors in $\mathbb{R}^n$, show that
$\underline{u} \cdot \underline{v} = \frac{1}{4}|\underline{u}+\underline{v}|^2 - \frac{1}{4}|\underline{u}-\underline{v}|^2$

Solution

This isn't really related to the preceeding material, but it is an identity that you need to have seen at some point in your mathematical careers.

It is easiest if we start with the right hand side and prove that it is equal to the left hand side

$$\frac{1}{4}|\underline{u}+\underline{v}|^2 - \frac{1}{4}|\underline{u}-\underline{v}|^2 = \frac{1}{4}|(\underline{u}+\underline{v})^2| - \frac{1}{4}|(\underline{u}-\underline{v})^2|$$

$$= \frac{1}{4}|\underline{u}\cdot\underline{u} + 2\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}| - \frac{1}{4}|\underline{u}\cdot\underline{u} - 2\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}|$$

$$= \frac{1}{4}\left[\underline{u}\cdot\underline{u} + 2\,\underline{u}\cdot v + \underline{v}\cdot\underline{v} - \underline{u}\cdot\underline{u} + 2\,\underline{u}\cdot\underline{v} - \underline{v}\cdot\underline{v}\right]$$

$$= \frac{1}{4}\left[4\underline{u}\cdot\underline{v}\right] = \underline{u}\cdot\underline{v}$$