$$\bar{z} = \frac{1}{8\pi}\int_0^4 \left[z\pi r^2\right]_0^{z^{1/2}} dz = \frac{1}{8\pi}\int_0^4 z^2\pi\, dz = \frac{1}{8\pi}\left[\frac{z^3\pi}{3}\right]_0^4$$

$$= \frac{1}{8\pi}\left(\frac{64\pi}{3}\right) = \frac{8}{3}.$$

Question 2

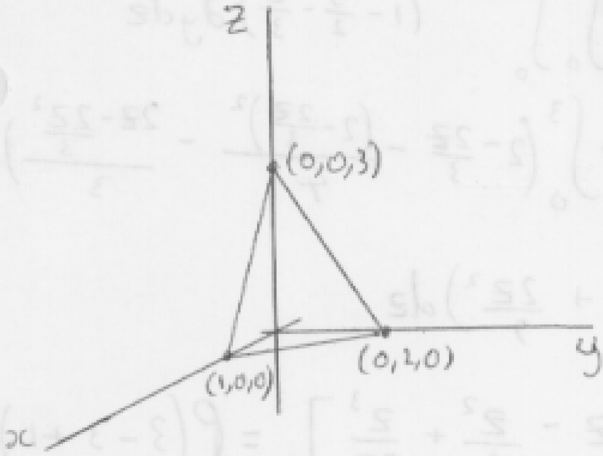
Find the mass of the tetrahedron of constant density cut from the positive octant by the coordinate planes and the plane $x + \frac{y}{2} + \frac{z}{3} = 1$.

Solution

Firstly, we must draw the tetrahedron, so that we can work out how to integrate over it.

The positive octant, refers to the three dimensional region in which the x, y and z coordinates are all positive. The coordinate planes are the planes on which lie two of the axis. So these conditions give us three of the four faces of the tetrahedron. Now, when $x=0$ and $y=0$, the plane $x+\frac{y}{2}+\frac{z}{3}=1$ tells us that $z=3$. When $x=0$ and $z=0$, $y=2$ and when $y=0$ and $z=0$, $x=1$, so the plane $x+\frac{y}{2}+\frac{z}{3}=1$ crosses the axis at the points $(0,0,3)$, $(0,2,0)$ and $(1,0,0)$.

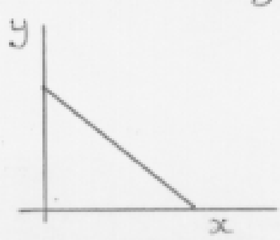
Now we have enough information to make a sketch of the tetrahedron.



The tetrahedron is the pyramid-like shape whose edges are the axis plus the three lines drawn in.

To work out the mass, we will take slices parallel to the x-y plane. The slices will be triangular, so we will stick to Euclidean coordinates throughout.

Each slice will look like



However, the size of the triangle will vary depending on whether the slice is taken from near the