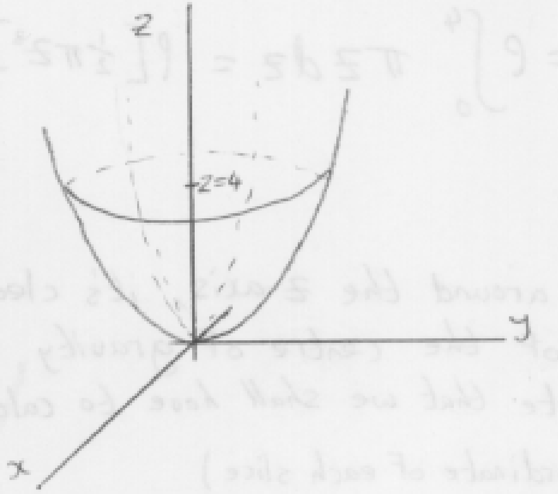
Tutorial 12

Question 1 Find the mass of the region of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Find also its centre of gravity.

Solution

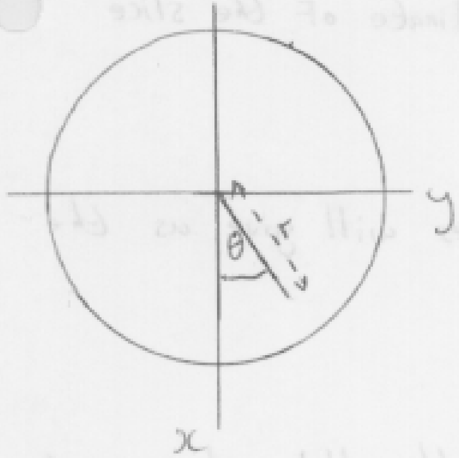
Firstly we need to draw the shape, in order to get an idea of how best to integrate it.



So the shape is like that of a full bowl.

Given the circular symmetry of the shape, around the z-axis. We would be best dividing the shape into circular slices, integrating of each slice using polar coordinates and summing up all the slices.

To get the area of one slice, we start with our infinitely small square $r\,d\theta\,dr$ and we sum over the slice $\iint r\,d\theta\,dr$. We are including the whole circle, so the limits on the θ-integration are 0 and 2π.



The radius of the slice is a function of the z coordinate. From the upper diagram, we can see that a slice taken from near the top of the shape will have a larger radius than a slice taken from lower down.

We determine the upper limit with the equation $z = x^2 + y^2$. We know that a circle is described by the equation $x^2 + y^2 = r^2$ and by comparing the two, we can see that the radius of the circumference of the slice is related to z by $r^2 = z$ or $r = z^{1/2}$. So this is our upper limit on r.

Area of slice $= \int_0^{z^{1/2}}\int_0^{2\pi} r\,d\theta\,dr$. Multiplying the slice by its thickness