$$\text{So } \bar{z} = \frac{\rho\int_0^4\int_0^{\sqrt{4-z}}\int_0^{2\pi} z r\,d\theta\,dr\,dz}{\rho\int_0^4\int_0^{\sqrt{4-z}}\int_0^{2\pi} r\,d\theta\,dr\,dz}$$

Evaluating this, $$\bar{z} = \frac{\int_0^4\int_0^{\sqrt{4-z}} 2\pi z r\,dr\,dz}{\int_0^4\int_0^{\sqrt{4-z}} 2\pi r\,dr\,dz} = \frac{\int_0^4 \pi z\,[r^2]_0^{\sqrt{4-z}}\,dz}{\int_0^4 \pi\,[r^2]_0^{\sqrt{4-z}}\,dz}$$

$$= \frac{\int_0^4 \pi z(4-z)\,dz}{\int_0^4 \pi(4-z)\,dz} = \frac{\int_0^4 (4z-z^2)\,dz}{\int_0^4 (4-z)\,dz} = \frac{\left[2z^2-\frac{1}{3}z^3\right]_0^4}{\left[4z-\frac{1}{2}z^2\right]_0^4}$$

$$\bar{z} = \frac{32-\frac{64}{3}}{16-8} = \frac{32-21\frac{1}{3}}{8} = \frac{10\frac{2}{3}}{8} = \frac{32/3}{8} = \frac{4}{3}$$

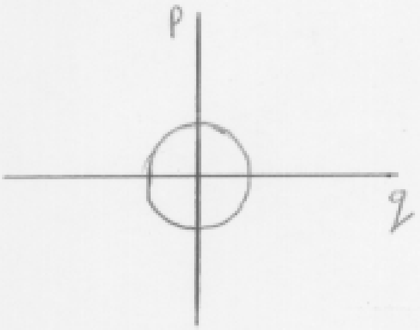
So the centre of mass lies at $(0,0,\frac{4}{3})$

Question 3

Evaluate $\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr$. (Note: here r does not denote radius).

Solution

We need to look at the shape of the region over which we are integrating, by looking at the limits.

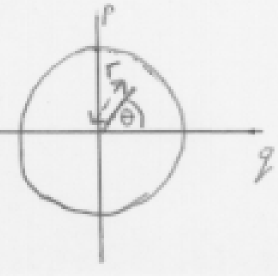


the limits on p are 0 to $\sqrt{4-q^2}$. $p=\sqrt{4-q^2}$ can be rearranged to get $p^2+q^2=4$, which is the equation of a circle of radius 2.

Because p only goes from 0 to $\sqrt{4-q^2}$, it excludes the half of the circle below the q-axis.

The q limits go from 0 to 2 which excludes everything to the left of the p axis. So we are only interested in the top right quarter of the circle. The r-integration goes from 0 to 7, which turns the circle into a cylinder of length 7. So the region over which we are integrating is a quarter of a cylinder.

Instead of using p and q coordinates, we will use polar coordinates instead.



We replace $dp\,dq$ with $r\,d\theta\,dr$. The limits on the r-integration are 0 to 2 because $p=\sqrt{4-q^2}$ is the equation for a circle of radius 2. The range of θ is 0 to $\pi/2$, because we are only concerned with a quarter of the circle. The one remaining term, which we need to