Now, what limits do we put on the integrals?

The inner most integral is in θ. We need to integrate over all 2π radians ($360°$) of each circular slice, ie we need to integrate over the whole circle, so the limits of θ integration are 0 and 2π

$$\text{Volume} = \int\int\int_0^{2\pi} r\,d\theta\,dr\,dz.$$

For the 2nd integral, which represents r-integration, the answer is slightly trickier. If we take a slice from the bottom of the bowl and a slice from the top and compare them, we can see that the radius of each slice is different. The slice from the top of the bowl will have a larger a radius than the one taken from near the bottom. So the upper limit of the r-integration must be dependent on z.

The sides of the bowl and hence the radius of each slice, is determined by the equation $z = x^2+y^2$. We also know that the equation of a circle is of the form $x^2+y^2=r^2$. By comparing the two equations, we can see that $r^2 = z$, ie the radius squared, of each slice, is equal to z. This meets our requirements because we have that $r = \sqrt{z}$.

$$\text{Volume} = \int\int_0^{\sqrt{z}}\int_0^{2\pi} r\,d\theta\,dr\,dz.$$

Finally, we need the limits to the z-integration. This is more straight forward, because we are the slices from the bottom of the bowl at $z=0$ to the top at $z=z_a$, so the limits are 0 to z_a. Giving,

$$\text{volume} = \int_0^{z_a}\int_0^{\sqrt{z}}\int_0^{2\pi} r\,d\theta\,d\ dz$$

Question 2

Find the centre of mass of the solid of constant density bounded below by the xy-plane and above by the paraboloid $z = 4-x^2-y^2$

Solution.

The shape is an upside down bowl shape and looks like