$\int \frac{1}{2} a^2(\theta)\, d\theta$. The strips start in the direction $-90°$ and finish in the direction $+90°$, so the limits on the integral are $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

So our integral expression is

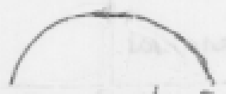
$$\int_{-\pi/2}^{\pi/2} \tfrac{1}{2} a^2(\theta)\, d\theta.$$

Question 2

Find the centre of mass of a semicircular wire of constant density and radius a.

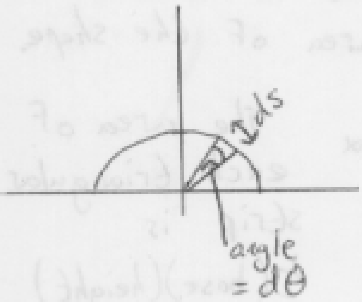
Solution

This question does not require us to find the area of the shape. However, we do need to use polar coordinates to answer it.



the semicircle of wire

The first thing to notice is that the shape is symmetric about the y-axis, so the x-component of the centre of mass lies at $\bar{x} = 0$. We only need to calculate the y-component.



To do this we divide the wire up into chunks, each ds long. If the density, in units of mass per length, is ρ, then the mass of the chunk is $\rho\, ds$.

Summing over all the chunks, to get the mass of the wire, we have $M = \sum \rho\, ds$. Letting ds become infinitely small, we replace $\sum$ with $\int$ to give us $M = \int \rho\, ds$. Now ds is just an infinitely small chunk of the wire, which is in the shape of an arc. We saw before that $ds = a\, d\theta$, so we can replace this in the integral, $M = \int \rho a\, d\theta$. If θ is the angle to the x-axis then, for the integral to include all the chunks of wire, θ must start at 0 and finish at π, which is $180°$, so our limits are $\int_0^{\pi}$ and the complete integral is $M = \int_0^{\pi} \rho a\, d\theta$. This is equal to $M = \pi a \rho$

To calculate the y-component of the centre of mass we use a fraction, in the numerator of which is a sum over all the chunks, ds, with each multiplied by its y-coordinate. In the denominator is the sum over all the chunks, to give the total mass. This is what we have just calculated. So, we have the numerator left to do.

The mass of each chunk is $\rho\, ds$. Multiplying this by its y-coordinate gives $y\rho\, ds$ and summing them up gives us $\int_0^{\pi} y \rho\, ds$