So instead of $\bar{x} = \dfrac{M_1 x_1 + M_2 x_2 + M_3 x_3}{M_1 + M_2 + M_3}$, we use $\bar{x} = \dfrac{\sum_{x=0}^{a} \rho F(x) s\, x}{\sum_{x=0}^{a} \rho F(x) s}$

The numerator is a sum over the mass of each strip times its position and the denominator is a sum over the mass of each strip.

So far, this is only an approximation, because the strips are finitely thick and some of their corners protrude above the curve and some fall below the curve. However, if we take the strips to be infinitely thin, then we get an exact expression for $\bar{x}$. Changing our notation to reflect this, we get

$$\bar{x} = \frac{\int_0^a \rho x F(x)\, dx}{\int_0^a \rho F(x)\, dx}$$

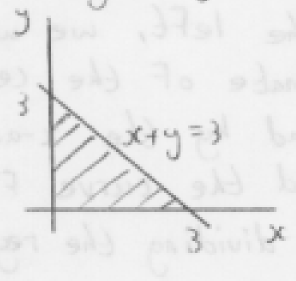
where ρ is the density, $F(x)$ the length of an infinitely thin strip located at x and dx is the infinitely small width of each strip.

Questions

1) Find the centre of mass of the regions of constant density
a) cut from the first quadrant by $x+y=3$
b) bounded below by the x-axis and above by $y=\sqrt{1-x^2}$

Solution

a) Firstly, we need to look at the region over which we are integrating. The first quadrant refers to the quarter of the x-y plane where x and y are both positive, so the shape must be the shaded triangular region in the diagram.



To work out the x-coordinate of the centre of gravity, we divide the region into strips that lie in the y-direction.

Each strip is dx wide. The length of each strip is obtained from the equation $x+y=3$, giving $y=3-x$. So the area of each strip is $(3-x)dx$. The mass of each strip is $\rho(3-x)dx$.

To obtain the numerator in $\bar{x}$, we need to multiply the mass of each strip by its position, giving $\rho(3-x)x\,dx$. We then need to sum over all the strips. So for the numerator, we have $\int_0^3 \rho(3-x)x\,dx$. For the denominator, we need to sum up just the mass of each strip, which gives $\int_0^3 \rho(3-x)\,dx$.