261: 25th October

Recap

We want to solve the matrix equation, $Ax = b$, where we know the matrix A and the vector b, but don't know the vector x. Last time, we saw that we could do this, if we knew how to decompose A into an upper triangular matrix, U, and a lower diagonal matrix, L.

For example, we want to solve the following for x.

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 3 \\ 6 & 12 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 23 \\ 57 \end{pmatrix}$$

If we know that A is the product of L and U,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we can say that $Ax = b$ is equivalent to $LUx = b$. Setting $Ux = y$, where y is another unknown vector, we then have $Ly = b$. This looks like

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 23 \\ 57 \end{pmatrix}$$

Using forward substitution, we can use the first line to get y_1, then use y_1 in the second line to obtain y_2 and then use both y_1 and y_2 to get y_3 from the third line. This gives us the vector y. Similarly, we can then use backward substitution to solve $Ux = y$ for x.

1) Frobenius matrices. So far, in each question you have been provided with the LU decomposition of the matrix A. The Frobenius matrix method is a way of calculating the L and U matrices for a given matrix A.

Lets say that A is now a 4*4 matrix and that each number in the matrix is represented by a variable labelled by its position in the matrix. This looks like

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

The idea is that when we multiply this by the first frobenius matrix, the product has zeros in the first column, beneath the first entry. Then, when we multiply by the second frobenius matrix, the new product has zeros beneath the first entry of the first column and beneath the second entry of the second column. Similary, when we multiply by the third frobenius matrix, the same pattern of zeros in the first two columns and a zero in the bottom place of the third row. This makes the product upper diagonal.

The first frobenius matrix looks like this

$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & 0 \\ -\frac{a_{31}}{a_1 1} & 0 & 1 & 0 \\ -\frac{a_{41}}{a_{11}} & 0 & 0 & 1 \end{pmatrix}$$

Now $F_1 A$ will be of this form,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^2 & a_{23}^2 & a_{24}^2 \\ 0 & a_{32}^2 & a_{33}^2 & a_{34}^2 \\ 0 & a_{42}^2 & a_{43}^2 & a_{44}^2 \end{pmatrix}$$

The numbers with the superscript ^{2} will be different from those of the original matrix A and will need to be worked out. Let's call this matrix A^2.

Now the second Frobenius matrix will look like this,

$$F_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a_{32}^2}{a_{22}^2} & 1 & 0 \\ 0 & -\frac{a_{42}^2}{a_{22}^2} & 0 & 1 \end{pmatrix}$$

Now $F_2 F_1 A$ will look like this

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^2 & a_{23}^2 & a_{24}^2 \\ 0 & 0 & a_{33}^3 & a_{34}^3 \\ 0 & 0 & a_{43}^3 & a_{44}^3 \end{pmatrix}$$

Where the numbers with the superscript ^{3} will have to be calculated.

We can continue, creating a third frobenius matrix, F_3, so that $F_3 F_2 F_1 A$ is upper diagonal. Let's call $F_3 F_2 F_1 A$, U.

Before we continue, try calculating the product of the following.

a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 3 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{2}{18} & -\frac{5}{18} & \frac{1}{6} \end{pmatrix}$$

You can see from this that the product of two lower diagonal matrices is also lower diagonal and that the inverse of a lower diagonal matrix is lower diagonal. The Frobenius matrices are all lower diagional and so their products and inverses must be lower diagonal.

Now our aim is to deconstruct A into the product of a lower and upper diagonal matrix. Obviously, the expression $F_1^{-1}F_2^{-1}F_3^{-1}F_3F_2F_1A$ is just going to be equal to A, because each of the Frobenius matrices is going to cancel out. However, the part $F_3F_2F_1A$ is the upper diagonal expression that we called U, above.

Now, if the product of lower diagonal matrices is lower diagonal and the inverse is too, then the expression $F_1^{-1}F_2^{-1}F_3^{-1}$ must be lower diagonal. If we call this expression L, then this means that we can express A as the product of L and U, ie $A = LU$.

This is how to LU decompose any matrix.

Have a go at LU decomposing the following. In each case, calculating F_1, F_2 and U is adequate. You don't need to calculate L or l^{-1}.

$$c) \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \qquad\qquad d) \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix}$$

2) Gauss-Jordan Elimination. Here we take the Frobenius method one step further. We start with the matrix equation, $Ax = b$ and apply the Frobenius matrices to both sides, giving $F_3F_2F_1Ax = F_3F_2F_1b$. We know that this makes the left hand side is upper diagonal. Now, in much the same way as we did before, we can eliminate the numbers above the leading diagonal by successively multiplying the equation by special matrices.

For the matrix,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

we can left-multiply it by

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & -\frac{a_{14}}{a_{44}} \\ 0 & 1 & 0 & -\frac{a_{24}}{a_{44}} \\ 0 & 0 & 1 & -\frac{a_{34}}{a_{44}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To get something that looks like

$$\begin{pmatrix} a_{11} & a_{12}^2 & a_{13}^2 & 0 \\ 0 & a_{22}^2 & a_{23}^2 & 0 \\ 0 & 0 & a_{33}^2 & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

where the superscript, again, ^{2} denotes new values to be calculated. [The values of a_{ij} in (2) aren't the same as those in (1)].

In much the same way as before, we can keep doing this until we arrive at a product that's completely diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22}^3 & 0 & 0 \\ 0 & 0 & a_{33}^2 & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Then if we divide each row of the matrix and each row of vector on the right, by the value of a on that row, then the vector on the right must be equal to x, ie

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}^3} \\ \frac{b_3}{a_{33}^2} \\ \frac{b_4}{a_{44}} \end{pmatrix}$$

This is Gaussian-Jordan elimination.

The same idea can be used to calculate the inverse of the original matrix A. Starting from the matrix equation $Ax = b$, we can insert the unit matrix infront of the b vector, without changing the meaning of the equation. This gives $Ax = Ib$. Then we apply the Frobenius matrices and the analogous G matrices to both sides of the equation. This makes the matrix on the left-hand side diagonal, but changes the matrix on the right hand side to a non-diagonal matrix. If we then divide each row of both matrices so that the left hand matrix becomes the identity, then the matrix on the right-hand side must be the inverse.

Let me illustrate this. We start with $Ax = Ib$ and apply the Frobenius and G matrices. $G_3G_2G_1F_3F_2F_1Ax = G_3G_2G_1F_3F_2F_1Ib$. The matrix on the left-hand side is now diagonal. If we divide each row of both matrices by the value of the left-hand matrix, on that row, then the left hand side becomes the identity. Let's say that this makes the matrix on the right-hand side equal to Y. Then this gives us $Ix = Yb$, which is that same as $x = Yb$. Hence, Y must be equal to the inverse of A.

Calculate F_1, G_1 and A^{-1} for each of the following

$$e) \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad\qquad f) \quad A = \begin{pmatrix} 2 & 5 \\ 6 & 1 \end{pmatrix}$$