261: 18th October

1) (Don't worry this question is mostly reading). Why do we need any tricks? Well here's an example of a standard technique used to calculate the inverse of a matrix. If you haven't seen this yet, don't worry it'll be introduced to you later on in the course.

We want to calculate the inverse of the following.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

To do so we associate the top row with the unit vector $(1, 0, 0)$, the middle with $(0, 1, 0)$ and the bottom with $(0, 0, 1)$. In effect, we put a second identity matrix alongside the first.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We then work out what linear combination of the rows (from matrix A) is needed to construct the row vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and perform the same operations on the row vectors from the identity matrix.

Using the row vectors from matrix A, we can get $(1, 0, 0)$ from 2*(third row)+(second row)-2*(first row). Performing the same operation on the row vectors of the identity matrix, 2*(third row)+(second row)-2*(first row) gives us $2(1, 0, 0) + (0, 1, 0) - 2(0, 0, 1) = (-2, 1, 2)$.

$(0, 1, 0)$ is equal to 3*(first row of matrix A) - 2*(second row of matrix A) - (third row of matrix A). Performing this on the row vectors of the identity matrix gives us $3(1, 0, 0) - 2(0, 1, 0) - (0, 0, 1) = (3, -2, -1)$.

$(0, 0, 1)$ is equal to (second row) - (first row). In terms of row vectors of the identity matrix, this is, $(0, 1, 0) - (1, 0, 0) = (-1, 1, 0)$.

We now have a correspondance between

$$\begin{aligned} (1, 0, 0) &\Leftrightarrow \quad (-2, 1, 2) \\ (0, 1, 0) &\Leftrightarrow \; (3, -2, -1) \\ (0, 0, 1) &\Leftrightarrow \quad (-1, 1, 0) \end{aligned}$$

If we put this in the form of a matrix with the left hadn matrix being the identity, then the right hand matrix will be inverse.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 2 \\ 3 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Now how many mathematical operations did we have to use to do this? Well lets start of my generalizing. Say we used an $n * n$ matrix instead of a $3 * 3$.

In general, to get each unit row vector on the left hand side, we would have to multiply each row by a number and and add them together. So, to construct one unit vector, we would have $n+1$ multiplications on each row (n terms in the matrix plus one nonzero term in the identity matrix). Over all the rows it would be $n(n+1)$ multiplications. There would also be $n(n+1)$ additions.

In total that is $n(n+1)$ multipliction operations and $n(n+1)$ addition operations and so far we've only calculated one unit row vector, so we have to multiply that by n again, for n unit vectors. If we use $\bigotimes$ to denote the multiplication operation and $\bigoplus$ to denote the addition operation that gives us $n(n(n+1)\bigotimes + n(n+1)\bigoplus)$.

Now a modern PC works at roughly 1GHz. Lets say that it can do one addition or one mulitplication in each cycle, ie 10^9 operations per second.

How long would it take to calculate the inverse of

a) a $100 * 100$ matrix (ie $n = 100$)?

b) a $1000 * 1000$ matrix ($n = 1000$)?

c) a $10,000 * 10,000$ matrix ($n = 10,000$)?

Believe it or not, a $n = 10,000$ matrix would not be unreasonable for an aerodynamics simulation. Given the answer you have for (c), you should be able to see why it is desirable to cut down the time, by taking advantage of any tricks available to you.

2) Cramer's rule. This method doesn't give you the inverse of a matrix, but instead calculates x from the equation $Ax = b$, where A is a known matrix and b a known vector. If you're only ever going to have to solve $Ax = b$ once, ie for one value of x and b, then this is an easier method that calculating A^{-1} directly. However, if you are going to have to solve $Ax = b$ several times for different b, then calculating A^{-1} is a better approach, because you'll only have to calculate it once and then getting x becomes just a matter of matrix-vector multiplication.

The method is as follows, for $Ax = b$, where A is the same matrix as above and b is $(14, 17, 6)$

1) Work out the determinant of A. For the same matrix as in question (1), the determinant is 1. We'll call this d.

2) Then replace the first column of A with the vector b and work out the determinant of the new matrix, ie

$$det \begin{pmatrix} 14 & 2 & 3 \\ 17 & 2 & 4 \\ 6 & 1 & 1 \end{pmatrix}$$

3) The value of the first component of x is then this value divided by d.

4) To the calculate the second component, we restore the first column to its original value and replace the second column with b, ie

$$\begin{pmatrix} 1 & 14 & 3 \\ 1 & 17 & 4 \\ 1 & 6 & 1 \end{pmatrix}$$

The determinent of this divided by d gives the second component. Similarly the third component is the determinent of A with the third column replaced, divided by d and so on.

Have a go at these.

a)

$$A = \begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

b)

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$$

c)

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ 6 \\ 3 \end{pmatrix}$$

3) Gaussian Elimination. Again we want to solve $Ax = b$, where we know A and b. Now suppose A is upper diagonal, ie everything beneath the leading diagonal is zero. We had something that looks like this.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 3 \end{pmatrix}$$

It's quite easy to solve for x now. If we treat it like simultaneous equations, we have:-

$$\begin{gathered} x_1 + 2x_2 + 3x_3 = 7 \\ 2x_2 + 4x_3 = 6 \\ x_3 = 3 \end{gathered}$$

So we can solve for x_3, then use x_3 to get x_2 and finally use both to get x_1. This is known as backward subsitution. A formula for it is

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} u_{ij} x_j}{u_{ii}}$$

Now suppose A is lower diagonal, ie everything above the lead diagonal is zero. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 3 \end{pmatrix}$$

As before we can consider this as a set of simultaneous equations

$$\begin{gathered} x_1 = 7 \\ x_1 + 2x_2 = 6 \\ x_1 + x_2 + x_3 = 3 \end{gathered}$$

and solve for x_1, followed by x_2 and then x_3. This is forward subsitution.

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} u_{ij}x_j}{u_{ii}}$$

Now for LU decomposition. Imagine that A is neither upper diagonal nor lower diagonal; it's just an ordinary matrix. If we could decompose it into the product of an upper diagonal matrix, U, and a lower diagonal matrix L, then we could say that $A = LU$ and $Ax = LUx$, so $LUx = b$.

Now, let's rename the vector Ux, y, then we have $Ly = b$. Now this is relatively straight forward, because we can use forward subsitution to solve for y. Then we could use backward substitution to solve $Ux = y$ and hey presto, we will have x.

Have a go at the following (ie solve for x)

a)

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 3 \\ 6 & 12 & 9 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 3 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ 23 \\ 57 \end{pmatrix}$$

b)

$$A = \begin{pmatrix} 2 & 12 & 2 \\ 1 & 10 & 3 \\ 1 & 10 & 6 \end{pmatrix} \quad L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 28 \\ 24 \\ 27 \end{pmatrix}$$