261:Numerical Integration
28th February

Last week concluded our study of Interpolation. This week we will start a new, but related topic, Numerical Integration.

The idea behind all the interpolation methods that we have seen is quite a simple one. Given a set of points, taken from some unknown function, we want to create a new funtion that approximates the unknown function, as accurately as possible.

However, the usefulness of interpolation does not necessarily end there. It would be quite reasonable to want to approximate the derivative or the integral of the unknown function. We can do this two ways. The first and more obvious way to do it, is to create an interpolating function and then either differentiate it or integrate it. The second is to construct the approximate integral or derivative directly from the given points.

We will start by looking at an example of the first approach.

The Newton-Cotes Scheme

The Newton-Cotes scheme takes the Lagrange Interpolating formula and integrates it.

We will start by having a look at the Lagrange Interpolating formulae. Suppose that we have $n+1$ support points (ie data points), $(x_0, f_0), (x_1, f_1), ..., (x_n, f_n)$. We can create an approximate function that looks like

$$P_n(x) = \sum_{i=0}^{n} f_i L_i(x)$$

as long as

$$L_i(x_j) = \delta_{ij}$$

This condition ensures that

$$P_n(x_j) = f_j$$

The $L_i(x)$ are known as the Lagrange polynomials and take the form

$$L_i(x) = \frac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_{n-2})(x-x_{n-1})}{(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_{n-2})(x_i-x_{n-1})}$$

We can see from this definition that when $x = x_{j \neq i}$, $L_i(x_j) = 0$, but that when $x = x_i$, $L_i(x_i) = 1$.

Let us try to integrate $P_n(x)$

$$\int_a^b P_n(x)dx = \int_a^b \sum_{i=0}^{n} f_i L_i(x)dx$$

$$\int_a^b P_n(x)dx = \sum_{i=0}^{n} f_i \int_a^b L_i(x)dx$$

So our definition of the approximate integral depends on the values of $\int_a^b L_i(x)dx$. In general, there is no way of simplifying these terms. Howver, we can remove their dependence on the limits a and b.

Let's write $L_i(x)$ as

$$L_i(x) = \prod_{k=0,k\neq i}^{n} \frac{x - x_k}{x_i - x_k}$$

If we define $h = \frac{b-a}{n}$ and a new coordinate t so that $x = a + ht$, then when $t = 0$, $x = a$ and when $t = n$, $x = b$.

Furthermore, if the x_i are evenly spaced and $a = x_0$ and $b = x_n$, then

$$\begin{aligned} L_i(x) &= \prod_{k=0,k\neq i}^{n} \frac{a + ht - a - hk}{a + hi - a - hk} \\ &= \prod_{k=0,k\neq i}^{n} \frac{ht - hk}{hi - hk} \\ &= \prod_{k=0,k\neq i}^{n} \frac{t - k}{i - k} \end{aligned}$$

We will call the last line $\phi_i(t)$ because it is only a function of t. Now if $x = a + ht$, then $\frac{dx}{dt} = h$ or $dx = hdt$

$$\int_a^b L_i(x)dx = \int_0^n \phi_i(t)hdt$$

These terms can't be simplified in a general fashion, so we label them $h\alpha_i$. Now our goal all along, has been to obtain an approximate expression for the integral of the unknown function that generated the support points. We obtain this expression by integrating the interpolating polynomial.

$$\begin{aligned} \int_a^b P_n(x)dx &= \int_a^b \sum_{i=0}^{n} f_i L_i(x)dx \\ &= h\sum_{i=0}^{n} f_i\alpha_i \end{aligned}$$

The weights, α_i have an interesting property.

$$\sum_{i=1}^{n} \alpha_i = n$$

which means that

$$h\sum_{i=0}^{n} \phi_i(t) = 1$$

We can prove this by seeing what happens when we assume that the unknown function is $f(x) = 1$. In this case, all $f_i = 1$ and we assume that the integral

$$\int_a^b P_n(x)dx$$

will be the same as

$$\int_a^b f(x)dx = b - a$$

Consequenly, if

$$\int_a^b P_n(x)dx = h\sum_{i=0}^{n} f_i\alpha_i$$

then we have that

$$b - a = h\sum_{i=0}^{n} \alpha_i$$

$$b - a = \frac{b-a}{n}\sum_{i=0}^{n} \alpha_i$$

So

$$n = \sum_{i=0}^{n-1} \alpha_i$$

Question 1) Use the Newton-Cotes scheme to find the integral based upon the following sets of points of the form (x_i, f_i)

a) $(1, 1)$, $(2, 1)$, $(4, 2)$

b) $(1, 0)$, $(4, 3)$

Extended Trapezoidal Rule

We shall now see that in the last question, we derived the extended trapezoidal rule from the Newton-Cotes scheme.

Suppose that we have $n + 1$ support points, but that instead of constructing and integrating P_n, we take pairs of consecutive points and use the Newton-Cotes scheme on each pair.

On the interval $[x_i, x_{i+1}]$

$$\alpha_i = \int_0^1 \frac{t-1}{-1}dt = \frac{1}{2}$$

$$\alpha_{i+1} = \int_0^1 \frac{t}{1}dt = \frac{1}{2}$$

$$h = \frac{x_{i+1} - x_i}{1}$$

So the integral over the interval is

$$h(f_i\alpha_i + f_{i+1}\alpha_{i+1})$$

$$= \frac{1}{2}(f_i + f_{i+1})(x_{i+1} - x_i)$$

This is just equal to the length of the base of the interval time the mean height.

If we add this up over all the support points, we have that

$$T(h) = \frac{1}{2}h(f_0 + f_1) + \frac{1}{2}h(f_1 + f_2) + \frac{1}{2}h(f_2 + f_3) + ... + \frac{1}{2}h(f_{n-1} + f_n)$$

$$= h(\frac{f_0}{2} + f_1 + f_2 + ... + f_{n-3} + f_{n-1} + \frac{f_n}{2}$$

This is the extended trapezoidal rule.

Question 2) Using the extended trapezoidal rule, calculate the integral of the following sets of support points of the form (x_i, f_i)

a) $(1, 0)$, $(2, 1)$, $(3, 4)$, $(4, 6)$

b) $(1, 2)$, $(2, 4)$, $(3, 5)$, $(4, 4)$, $(5, 3)$