261: 21st February

Interpolation continued

Recap

Over the past couple of weeks, we have looked at various schemes for polynomial interpolation. In each, we started with a polynomial of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$ and looked at ways of choosing the coefficients $a_0, ..., a_n$ so that it fitted the data.

However, the failings in these schemes become apparent when we try to interpolate data points that have been derived from functions of the form $\frac{1}{2x}$ or $\frac{5x}{1-2x+12x^7}$. If we were to try to use the polynomial methods, we would certainly be able to reproduce all the data points (as long as they were finite), but the polynomial wouldn't even come close to reproducing any poles that might exist in the original equations. So, although there is nothing to stop us using the polynomial methods, it would be make more sense to start with a function of the form

$$\frac{a_0 + a_1x + a_2x^2 + ... + a_\mu x^\mu}{b_0 + b_1x + b_2x_2 + ... + b_\nu x^\nu}$$

This is exactly what Rational function interpolation does.

Rational function interpolation

We start with the rational function

$$\Phi^{\mu\nu}(x) = \frac{a_0 + a_1x + a_2x^2 + ... + a_\mu x^\mu}{b_0 + b_1x + b_2x_2 + ... + b_\nu x^\nu}$$

in which there are $\mu + \nu + 2$ coefficients, $a_0, ..., a_\mu$ and $b_0, ..., b_\nu$. Because $\Phi^{\mu\nu}$ is a fraction, we can multiply the top and the bottom by the same constant number, without changing the value of $\Phi^{\mu\nu}$. This means that there isn't a unique set of coefficients that will reproduce a set of data points. All we can hope to do is use the data points to work out the relationships between the coefficients.

Now, let's say that we pick an arbitrary value for b_0. How many data points do we need to work out the rest of the coefficients? The answer is $\mu + \nu + 1$ because this is how many unknown coefficients remain.

The data points, (x_i, f_i), are also known as *support* points and we can use them to calculate the coefficients as follows.

We need $\Phi^{\mu\nu}(x)$ to satisfy

$$\Phi^{\mu\nu}(x_i) = f_i$$

So

$$a_0 + a_1x_i + a_2x_i^2 + ... + a_\mu x_i^\mu = f_i(b_0 + b_1x_i + b_2x_i^2 + ... + b_\nu x_i^\nu)$$

or

$$a_0 + a_1x_i + a_2x_i^2 + ... + a_\mu x_i^\mu - f_i(b_0 + b_1x_i + b_2x_i^2 + ... + b_\nu x_i^\nu) = 0$$

There is one of these equations for each support point, (x_i, f_i), giving us a set of $\mu + \nu + 1$ simultaneous equations and $\mu + \nu + 2$ unknowns. This system of equations is denoted $S^{\mu\nu}$.

We can solve the simultaneous equations to get each coefficient in terms of b_0 which gives us our solution.

Now, constant numbers are not the only things by which we can multiply the top and bottom of $\Phi^{\mu\nu}$, without changing its value. We can also multiply by functions of x. When we have a rational function that contains no factors common to both the top and bottom, then $\Phi^{\mu\nu}$ can be described as *relatively prime*.

For each set of data points, there is only one relative prime, rational function that describes them. If we are presented with two different rational functions that describe the same points, then they must have the same relative prime.

Unfortunately, there is a big failing in the rational function method. Not every set of points can be recreated in a rational function. Imagine that we were trying to use the method to recreate 9 points taken from the function $x^8 + 2x^5 - x + 2$. According to our scheme, we could choose the polynomial

$$\Phi^{\mu\nu}(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}$$

However, because the highest order term in the numerator is of order 4, we wouldn't be able to recreate the polynomial $x^8 + 2x^5 - x + 2$. Whatsmore, we may not be able to come up with any polynomial that recreated all 9 points.

Question 1) Calculate the rational fucntions of the following data points. One of the three can't be interpolated using a rational function. Which one is it?

a)$(0, 0)$, $(1, \frac{3}{4})$, $(2, 1)$
b)$(0, 4)$,$(1, 1)$,$(2, 2)$
c)$(0, 6)$, $(1, 3)$,$(2, 2)$

Splines

The last interpolation method that we will look at is the spline method. In this scheme, we define a separate third order polynomial on the each interval between pairs of consecutive points. By insisting that the first and second derivatives match across each point, we get all but two of the equations needed to specify the values of the coefficients, in all the polynomials. We can get the last two by specifying the derivates at first and last data points.

Our starting point is to linearly interpolate between the data points. This means that in the interval $[x_i, x_{i+1}]$, we have the function

$$y(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} y_i + \frac{x - x_i}{x_{i+1} - x_i} y_{i+1}$$

Now, we want to modify this function so that its second derivative is linearly interpolated across the interval, too. As it stands, the second derivative, $y''(x)$, is zero, so we are free to add terms to it that linearly interpolate the second order. However, we have to be careful to choose these terms so that they don't change the value of $y(x)$ at x_i and at x_{i+1}.

We add to $y(x)$ the term

$$Cy''_i + Dy''_{i+1}$$

To make the notation neater, we will define

$$A = \frac{x_{i+1}-x}{x_{i+1}-x_i}, \quad B = \frac{x-x_i}{x_{i+1}-x_i}$$

With this definition, C and D are equal to

$$C = \frac{1}{6}(A^3 - A)(x_{i+1} - x_i)^2 \quad D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2$$

There are two important things to note about C and D. The first is that both are equal to zero, when $x = x_i$ and when $x = x_{i+1}$. This means that the extra terms don't interfere with the value of $y(x)$ at the points $x = x_i$ and $x = x_{i+1}$.

The second thing to notice is that

$$\frac{d^2C}{dx^2} = A \quad \frac{d^2D}{dx^2} = B$$

This means that

$$\frac{d^2y(x)}{dx^2} = Ay''_j + By''_{j+1}$$

A and B are the linear interpolation functions, so we have succeeded in creating a function that recreates the data points and interpolates linearly between the second derivatives.

$$y(x) = Ay_i + By_{i+1} + Cy''_i + Dy''_{i+1}$$

Unfortunately, this isn't a complete solution because we don't know the values of y''_i and y''_{i+1}. However, it turns out that if we insist that the first derivatives match across the intervals, then we can calculate the y''_i and y''_{i+1}.

To illustrate this

$$\frac{dy(x)}{dx} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{3A^2 - 1}{6}(x_{i+1} - x_i)y''_i + \frac{3B^2 - 1}{6}(x_{i+1} - x_i)y''_{i+1}$$

If we look at the polynomial in the preceeding interval, $[x_{i-1}, x_i]$, we will get a similar equation

$$\frac{dy(x)}{dx} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} - \frac{3A^2 - 1}{6}(x_i - x_{i-1})y''_{i-1} + \frac{3B^2 - 1}{6}(x_i - x_{i-1})y''_i$$

In the interval $[x_i, x_{i+1}]$, setting $x = x_i$ gives $A = 1$ and $B = 0$. In the preceeding interval, $[x_{i-1}, x_i]$, setting $x = x_i$ gives $A = 0$ and $B = 1$. If we insert these values and equate both expressions for $\frac{dy}{dx}$ we get

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{3A^2 - 1}{6}(x_{i+1} - x_i)y''_i + \frac{3B^2 - 1}{6}(x_{i+1} - x_i)y''_{i+1}$$

$$= \frac{y_i - y_{i-1}}{x_i - x_{i-1}} - \frac{3A^2-1}{6}(x_i - x_{i-1})y''_{i-1} + \frac{3B^2-1}{6}(x_i - x_{i-1})y''_i$$

becomes

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{2}{6}(x_{i+1} - x_i)y''_i - \frac{1}{6}(x_{i+1} - x_i)y''_{i+1}$$

$$= \frac{y_i - y_{i-1}}{x_i - x_{i-1}} + \frac{1}{6}(x_i - x_{i-1})y''_{i-1} + \frac{2}{6}(x_i - x_{i-1})y''_i$$

Rearranging this gives

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}} =$$

$$\frac{1}{3}(x_i - x_{i-1})y''_i + \frac{1}{6}(x_i - x_{i-1})y''_{i-1} + \frac{1}{3}(x_{i+1} - x_i)y''_i + \frac{1}{6}(x_{i+1} - x_i)y''_{i+1}$$

Which reduces to

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}} =$$

$$\frac{1}{3}(x_{i+1} - x_{i-1})y''_i + \frac{1}{6}(x_i - x_{i-1})y''_{i-1} + \frac{1}{6}(x_{i+1} - x_i)y''_{i+1}$$

On its own, this looks no easier to solve. However, we know everything in the equation except the $y''_i(x)$ and when we consider that there are $n-2$ of these equations (for n data points) it becomes apparent that we can represent these equations as one matrix equation.

If we label

$$d_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

$$a_{i,i} = \frac{1}{3}(x_{i+1} - x_{i-1})$$

$$a_{i,i-1} = \frac{1}{6}(x_i - x_{i-1})$$

$$a_{i,i+1} = \frac{1}{6}(x_{i+1} - x_i)$$

Then we have that

$$d_i = a_{i,i-1}y''_{i-1} + a_{i,i}y''_i + a_{i,i+1}y''_{i+1}$$

We must be cautious when applying this to the extremes of the range of our data points. If our points start with (x_0, y_0), then we will need to modify, somehow, the equation that refers to x_{-1} and y''_{-1} because they don't exist as a member of our data set. Equally, if the last point is at (x_{n-1}, y_{n-1}), the equation that refers to x_n and y''_n requires special attention.

There are three solutions to this problem. The first is simply to assume that y''_{-1} and y''_n are both equal to zero. This has the effect of setting the terms containing x_{-1} and x_n to zero. The second is to assume periodic boundary conditions and the third is to create some values for the extreme points, thereby deliberately fixing the values of $\frac{dy}{dx}$ at both extremes.

Using the first solution, the matrix looks like

$$\begin{pmatrix} a_{0,0} & a_{0,1} & 0 & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & 0 \\ \hdotsfor{6} \\ 0 & \cdots & 0 & a_{n-2,n-3} & a_{n-2,n-2} & a_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & a_{n-1,n-2} & a_{n-1,n-1} \end{pmatrix} \begin{pmatrix} y''_0 \\ y''_1 \\ y''_2 \\ \cdots \\ \cdots \\ y''_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \cdots \\ \cdots \\ d_{n-1} \end{pmatrix} which we can solve, using standard matrix methods.$$