261: 14th February

Interpolation continued

Recap

In the last tutorial, we looked at two ways to construct polynomials that recreate a given set of data points and that predict realistic values in the intervals between the data points.

The first method associated a Lagrange polynomial with each data point. The Lagrange polynomials acted like delta functions, so that when x coincided with one of the data points, the others contributed nothing to the value of the polynomial.

The second method was Neville's algorithm, which was an iterative approach. In it, we interpolated linearly between each pair of points and then to the resulting equations, interpolated lineraly between them. By repeating this process, of interpolating the equations that were generated at the previous step, we could arrive at a grand interpolating equation of sufficiently high order.

This week we will look at some more methods for interpolating.

Newton's divided difference scheme

In this scheme, we start by assumming that the polynomial must be of the form

$$P(x) = a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_3(x-x_0)(x-x_1)(x-x_2)+...+a_k(x-x_0)...(x-x_{k-1})$$

In order to have the same number of unknown coefficients as data points, we use the above polynomial when we have $k+1$ data points, $(x_0, f_0), (x_1, f_0), ..., (x_k, f_k)$.

To work out the values of the coefficients, we must be aware that, in the expression for $P(x_i)$, all the terms containing a_j, where $j > i$, will be equal to zero.

Using this, we can work out a_0 by setting x to x_0. At $x = x_0$

$$P(x_0) = a_0$$

So we set $a_0 = f_0$, so that

$$P(x_0) = f_0$$

Next we work out a_1. Setting $x = x_1$

$$P(x_1) = a_0 + a_1(x_1 - x_0)$$

If we require that $P(x_1) = f_1$, then

$$f_1 = a_0 + a_1(x_1 - x_0)$$

$$a_1 = \frac{f_1 - a_0}{x_1 - x_0}$$

For a_2, we set $P(x_2) = f_2$

$$P(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2 = \frac{f_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

and so forth.

There is an easier and computationally more efficient way to calculate the coefficients.

Following the same logic as above, we set $x = x_0$ and subsequently obtain that $a_0 = f_0$. We also get that

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

This method requires us to calculate similar quantities for adjacent pairs of data points, so here we will introduce the terms g_{ij}.

$$g_{01} = \frac{f_1 - f_0}{x_1 - x_0}$$

Similarly

$$g_{12} = \frac{f_2 - f_1}{x_2 - x_1}$$

$$g_{23} = \frac{f_3 - f_2}{x_3 - x_2}$$

etc. Now we obtain a_2 from

$$a_2 = \frac{g_{12} - g_{01}}{x_2 - x_1}$$

Let's prove that this is the same as the expression that we had for a_2 above.

$$a_2 = \frac{g_{12} - g_{01}}{x_2 - x_0}$$

$$a_2 = \frac{1}{x_2 - x_0}\left[\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{x_2 - x_0}\left[\frac{f_2}{x_2 - x_1} - \frac{f_1}{x_2 - x_1} + \frac{f_0}{x_2 - x_1} - \frac{f_0}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{x_2 - x_0}\left[\frac{f_2 - f_0}{x_2 - x_1} - \frac{f_1}{x_2 - x_1} + \frac{f_0}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}\left[(f_2 - f_0) - f_1 + f_0 - \frac{(f_1 - f_0)(x_2 - x_1)}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}\left[(f_2 - f_0) + \frac{(f_0 - f_1)(x_1 - x_0)}{x_1 - x_0} - \frac{(f_1 - f_0)(x_2 - x_1)}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}\left[(f_2 - f_0) + \frac{f_0x_1 - f_0x_0 - f_1x_1 + f_1x_0 - f_1x_2 + f_0x_2 + f_1x_1 - f_0x_1}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}\left[(f_2 - f_0) + \frac{f_1x_0 - f_1x_2 + f_0x_2 - f_0x_0}{x_1 - x_0}\right]$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)} \left[(f_2 - f_0) - \frac{(f_1 - f_0)(x_2 - x_0)}{x_1 - x_0} \right]$$

$$a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)} \left[(f_2 - a_0) - a_1(x_2 - x_0) \right]$$

Which is the same as our earlier definition.

Now we construct the terms g_{012}, g_{123}, etc as follows

$$g_{012} = \frac{g_{12} - g_{01}}{x_2 - x_0} \quad g_{123} = \frac{g_{23} - g_{12}}{x_3 - x_1}$$

so that

$$a_3 = \frac{g_{123} - g_{012}}{x_3 - x_0}$$

In general

$$a_i = \frac{g_{1..i} - g_{0..(i-1)}}{x_i - x_0}$$

Question 1) Calculate the coefficients a_0 to a_4 for the five data points of the form $(x_i,\ f_i)$: (1, 0.54), (2, -0.42), (3, -0.99), (4, -0.65), (5, 0.28). Plot the curve and see how closely it replicates a cosine curve.