Lecture notes for MA2326-2014

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Contents

1. Introduction

1.1. Basic Examples and Definitions.

Example The Harmonic Oscillator has the following equation

(1) $$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

which can also be written as

$$x''(t) + \omega^2 x = 0$$

where x is the dependent variable, t is the independent variable and ω is the parameter. We differentiate dependent variables with respect to independent ones, and anything left over is a parameter.

DEFINITION 1.1. *A differential equation is called ordinary if there is one independent variable*

It is called partial if there is more than one independent variable. A scalar equation or system of equations has one dependent variable. A vector equation/system has more.

Usually, number of dependent variables equals the number of equations.

Example Maxwell's Equations have 6 dependent variable and 8 independent variables.

DEFINITION 1.2. *The order of a differential equation is the order of the highest derivative.*

Eg, $\frac{d^2x}{dt^2} + \omega^2 x = 0$ has Order 2.

Example

(2) $$\frac{dx}{dt} - v = 0, \frac{dv}{dt} + \omega^2 x = 0$$

which is a first order system, with dependent variables x and v, independent variable t and parameter 1. This system is equivalent to the harmonic oscillator.

If x and v are solutions to (2) then x is a solution of (1). Similarly, if x is a solution of (1) then $x, \frac{dv}{dt} = v$ are solutions of (2).

Any differential equation or system is equivalent to a first order system or equation. This can be done by introducing extra dependent variables for lower derivatives, known as reduction of order. It is used mostly in proving theorems.

DEFINITION 1.3. *A linear equation or system is a linear equation (in the algebraic sense) in derivatives of dependent variables and the coefficients are functions of independent variables and parameters, not dependent variables.*

Example Legendre

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + v(v+1)y = 0$$

is a second order linear scalar, dependent variable y, independent variable x, and parameter v. It is a linear equation in $\frac{d^2y}{dx^2}, \frac{dy}{dx}, y$, with coefficients $1-x^2, -2x, v(v+1)$. The Harmonic Oscillator is linear in $\frac{d^2x}{dx^2}, \frac{dx}{dt}, x$ with coefficients $1, 0, \omega^2$.

DEFINITION 1.4. *An equation is called linear constant coefficient if it's coefficients are functions of the parameters, not dependent or independent variables.*

Independent variables and parameters will be real in this course, dependent variables are usually real.

Example Non Linear: Pendulum

$$\frac{d^2\theta}{d\theta^2} + \sin\theta = 0$$

For small $\theta, \sin\theta \approx \theta$ so

$$\frac{d^2\theta}{d\theta^2} + \theta \approx 0$$

We hope both solutions are similar to those for the harmonic oscillator with $\omega = 1$

2. Linear Equations

2.1. Homogeneous, Inhomogeneous, Existence and Uniqueness.

2.1.1. *Homogeneous, Inhomogeneous.* Homogeneous means all terms are of degree d. Inhomogeneous means all terms are of at most degree d.
Harmonic Oscillator $\frac{d^2x}{dt^2} + \omega^2 x = 0$ is a Linear Homogeneous Equation.
Forced Harmonic Oscillator $\frac{d^2x}{dt^2} + \omega^2 x = A\cos(vt + \phi)$ is an inhomogeneous linear equation.
$ax = b$ is homogeneous while $ax = 0$ is inhomogeneous.

THEOREM 2.1. *The set of all solutions of a linear homogeneous differential equation is a vector space*

PROOF. $S =$ the set of functions x satisfying $x'' + \omega^2 x = 0$ Suppose $x_1, x_2 \in S \Rightarrow x_1'' + \omega^2 x_1 = 0$ $x_2'' + \omega^2 x_2 = 0$
Let $x = c_1x_1 + c_2x_2$ Then $x' = c_1x_1' + c_2x_2'$ and $x'' = c_1x_1'' + c_2x_2''$.
So $x'' + \omega^2 x = c_1(x_1;; + \omega^2 x_1 + c_2(x_2 + \omega^2) = 0$
$\Rightarrow x \in S$, with $basis = \{\cos(\omega t),\ \sin(\omega t)\}$ Suppose x_1, x_2 satisfy $x'' + \omega^2 x = f(x)$. Then $x_1 - x_2 = x$ satisfies $x'' + \omega^2 x = 0$, where x'' is the corresponding homogeneous equation.

$$x_1'' + \omega^2 x_1 = f(x),\ x_2'' + \omega^2 x_2 = f(x)$$

$$x = x_1 - x_2,\ x' = -x_1' + x_2',\ x'' = x_1'' + x_2''$$

$$\Rightarrow x'' + \omega^2 x = x_1'' + \omega^2 x_1 - x_2'' - \omega^2 x_2$$

$$= f(x) - f(x) = 0$$

□

The general solution to the inhomogeneous problem is any particular solution plus a solution to the corresponding homogeneous problem.

2.1.2. *Existence, Uniqueness.* Given an ordinary differential equation and conditions, we can ask

(1) Existence-Is there a solution?
(2) Uniqueness-Is there at most one?
(3) What is the solution? Explicitly? Numerically?
(4) How regular is the solution? Is it differentiable?
(5) What is the limiting behaviour?
(6) How do the solutions depend on parameters? Conditions?
(7) Are there invariants of the equation?

Example Invariants $\frac{d^2x}{dt^2} + \omega^2 x = 0$

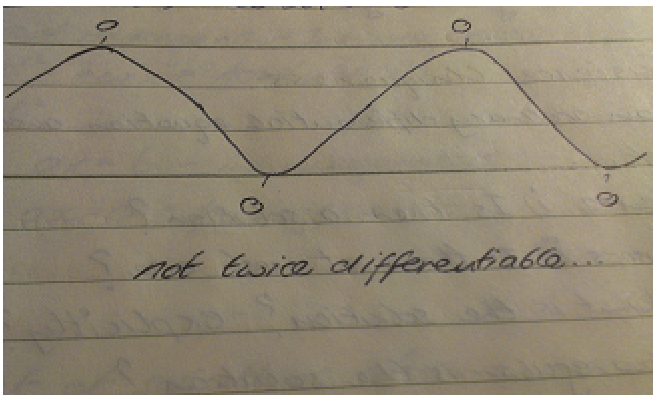
$$I = (\frac{dx}{dt})^2 + \omega^2 x^2$$

$$\frac{dI}{dt} = 2(\frac{dx}{dt})(\frac{d^2x}{dt}) + 2\omega^2 x\frac{dx}{dt} = 2\frac{dx}{dt}(\frac{d^2x}{dt^2} + \omega^2 x) = 0$$

$\Rightarrow I$ is locally constant for solutions.

THEOREM 2.2. *If $x : \mathbb{R} \to \mathbb{R}$ satisfies $x'' + \omega^2 x = 0$ then $I(t) = x'(t)^2 + \omega^2 x(t)^2$ is constant.*

THEOREM 2.3. *If $x : \mathbb{R} \to \mathbb{R}$ is a twice differentiable (continuous) function satisfying $x'(t)^2 + \omega^2 x(t)^2 =$ constant*
Then $x''(t) + \omega^2 x(t) = 0$ or x is a constant.



2.2. First Order Scalar Equations. Of the form

$$F(x,y,\frac{dy}{dx})=0$$

Take $\frac{dy}{dx}=xy$ say.
Formally, $\int\frac{dy}{y}=\int xdx$, integrating from (x_0,y_0)
$\ln(y)-\ln(y_0)=\frac{x^2}{2}-\frac{x_0^2}{2}\ \Rightarrow ln(y)-\frac{x^2}{2}=\ln(y_0)-\frac{x_0^2}{2}$ =constant.
$ye^{-\frac{x^2}{2}}=c$ a constant. $\Rightarrow y=ce^{\frac{x^2}{2}}$ and $c=y_0e^{-\frac{x_0^2}{2}}$
We need to prove:

(1) $y(x)=ce^{\frac{x^2}{2}},\ c=y_0e^{-\frac{x_0^2}{2}}$ solves $\frac{dy}{dx}=xy$ and $y(x_0)=y_0$
(2) If y solves $\frac{dy}{dx}=xy,\ y(x_0)=y_0$ then $y(x)=ce^{\frac{x^2}{2}},\ c=y_0e^{-\frac{x_0^2}{2}}$

PROOF. (a)

$$y(x)=ce^{\frac{x^2}{2}}\ \Rightarrow \frac{dy}{dx}=ce^{\frac{x^2}{2}}x=xy$$
$$y(x_0)=ce^{\frac{x_0^2}{2}}=y_0e^{-\frac{x_0^2}{2}}e^{\frac{x_0^2}{2}}=y_0$$

(b) $\frac{dy}{dx}=xy,\ y(x_0)=y_0$ Let $z(x)=y(x)e^{-\frac{x^2}{2}}$ Then

$$\frac{dz}{dx}=\frac{dy}{dx}e^{-\frac{x^2}{2}}+ye6-\frac{x^2}{2}(-x)=e^{-\frac{x^2}{2}}(\frac{dy}{dx}-xy)=0$$

$\frac{dz}{dx}=0\ \Rightarrow z$ is constant. So $z(x)=z(x_0)$.
$\Rightarrow y(x)e^{-\frac{x^2}{2}}=y_0e^{-\frac{x_0^2}{2}}=c$ so $y(x)=ce^{\frac{x^2}{2}}$

□

Checking that a given solution works is (almost) always purely mechanical. Will normally skip this step.
To prove uniqueness look for invariants.

Example $\frac{dy}{dx}=x^2y^2$
Formally, $\int\frac{dy}{y^2}=\int x^2dx$

$$-\frac{1}{y}+\frac{1}{y_0}=\frac{x^3}{3}-\frac{x_0^3}{3}$$
$$\Rightarrow \frac{1}{y}+\frac{x^3}{3}=\frac{1}{y_0}+\frac{x_0^3}{3}=c\ (const)$$
$$\frac{1}{y}=c-\frac{x^3}{3}\ \Rightarrow y=\frac{1}{c-\frac{x^3}{3}}$$

Which has a singularity when $x=\sqrt[3]{3c}$. Elsewhere, $y(x)=\frac{1}{c-\frac{x^3}{3}}$ is a solution of $\frac{dy}{dx}=x^2y^2$ which can easily be checked (if bored).
Are there any other solutions?
Take $z(x)=\frac{1}{y(x)}=\frac{x^3}{3}$ Then $\frac{dz}{dx}=-\frac{\frac{dy}{dx}}{y^2}+x^2=-\frac{x^2y^2}{x^2}=0$ so z is constant. $\Rightarrow z(x)=z(x_0)=c\ \Rightarrow \frac{1}{y}+\frac{x^3}{3}=c\ \Rightarrow y=\frac{1}{c-\frac{x^3}{3}}$ and $\frac{1}{y_0}+\frac{x_0^3}{3}=c$ So $y=\frac{1}{c-\frac{x^3}{3}}$ are the only solutions to $\frac{dy}{dx}=x^2y^2$ Or are they?
Exception:$y=0$ is a solution. In the definition of z we divide by y which could be 0. Division by 0 can lose solutions. Why did Ex.1 work then?
$y(x)=ce^{\frac{x^2}{2}}=0$ if $y_0=0$, so we just got lucky.

2.2.1. *Separable Equations.* Suppose $\frac{dy}{dx}=\frac{M(x)}{N(y)}$ where M,N are continuous. Also, assume for now that N has no zeroes.
In $\frac{dy}{dx}=xy,\ M(x)=x,\ N(y)=\frac{1}{y}$ and similarly in $\frac{dy}{dx}=x^2y^2,\ M(x)=x^2,\ N(y)=\frac{1}{y^2}$ In general:
$\frac{dy}{dx}=\frac{M(x)}{N(y)}\ \Rightarrow \int N(y)dy=\int M(x)dx$ integrating from (x_0,y_0). By the Fundamental Theorem of Calculus (FTC), $\exists\varphi,\ \psi$ with $\varphi'=M,\ \psi'=N$

$$\chi(y)-\chi(y_0)=\phi(x)-\phi(x_0)\ \Rightarrow\ \psi(y)-\varphi(x)=\psi(y_0)-\varphi(x_0)=const.$$

ψ is strictly monotone, so it is invertible and this equation can be solved for y $I(x)=\psi(y(x))-\varphi(x)$ should be invariant. $\psi(y(x))=\varphi(x)+C$, where $C=I(x_0)$ ψ is invertible, so $\exists\eta\ s.t\ \eta\circ\psi=\psi\circ\eta=id$

$$\Rightarrow \eta(\psi(y(x)))=\eta(\varphi(x)+c)=y(x)$$

Claim:

(1) $y(x) = \eta(\varphi(x) + c)$ is a solution to $\frac{dy}{dx} = \frac{M(x)}{N(y)}$
(2) If $\frac{dy}{dx} = \frac{M}{N}$ then $y(x) = \eta(\varphi(x) + c)$ for some c.

PROOF. (1) $y'(x) = \eta'(\varphi(x) + c)\varphi'(x)$ Also, $(\varphi' \circ \eta)\eta' = 1 \Rightarrow \eta' = \frac{1}{\psi' \circ \eta}$

$$y'(x) = \frac{1}{\psi'(\eta(\varphi(x) + c))}\varphi'(x) = \frac{M(x)}{N(y(x))}$$

(2)

$$y'(x) = \frac{M(x)}{N(y(x))}$$

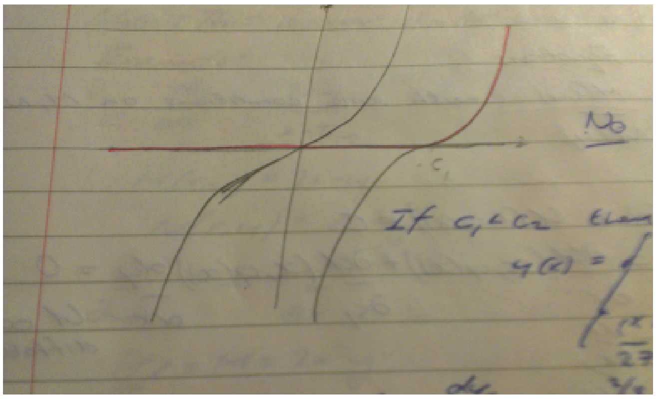
$I(x) = \psi(y(x)) - \varphi(x)$ and $c = \psi(y_0) - \varphi(x_0)$ Then $I'(x) = \psi'(y(x))y'(x) - \varphi(x) = N(y(x))y'(x) - M(x) = 0$ So $I(y(x)) = c$ i.e $\psi(y(x)) - \varphi(x) = c \Rightarrow \psi(y(x)) = \varphi(x) + c$ Then,

$$\eta(\psi(y(x))) = \eta(\varphi(x) + c) = y(x)$$

□

Note: Equations of the form $\frac{dy}{dx} = P(x)Q(y)$ are equivalent to the separable equation $\frac{dy}{dx} = \frac{M(x)}{N(y)}$ with $M = P,\ N = \frac{1}{Q}$ away from zeroes of Q, where we get constant solutions $y(x) = y_0$

Example $\frac{dy}{dx} = y^{\frac{2}{3}} = \frac{M(x)}{N(y)}$ with $M(x) = 1,\ N(y) = y^{-\frac{2}{3}}$ Then $\varphi(x) = x,\ \psi(y) = 3y^{\frac{1}{3}} \Rightarrow z = 3y^{\frac{1}{3}}\ \frac{z}{3} = y^{\frac{1}{3}}$
$\Rightarrow \frac{z^3}{27},\ \eta(z) = \frac{z^3}{27}$ so $y(x) = \eta(\varphi(x) + c),\ y(x) = \frac{(x+c)^3}{27}$ and can check that this works. Are these all the solutions to $\frac{dy}{dx} = y^{\frac{2}{3}}$?
No, $y(x) = 0$ is also a solution. So $y(x) = \{\frac{(x+c)^3}{27},\ 0\}$ are these all the solutions? No



If $c_1 < c_2$ then

$$f(x) = \begin{cases} \frac{(x+c_2^3)}{27} & x \leq -c_2 \\ 0 & -c_2 \leq x \leq -c_1 \\ \frac{(x+c_1)^3}{27} & x \geq -c_1 \end{cases}$$

Which is a solution to $\frac{dy}{dx} = y^{\frac{2}{3}}$ which is obvious except for $x = -c_2,\ x = -c_1$.
Derivative at $-c_1$:

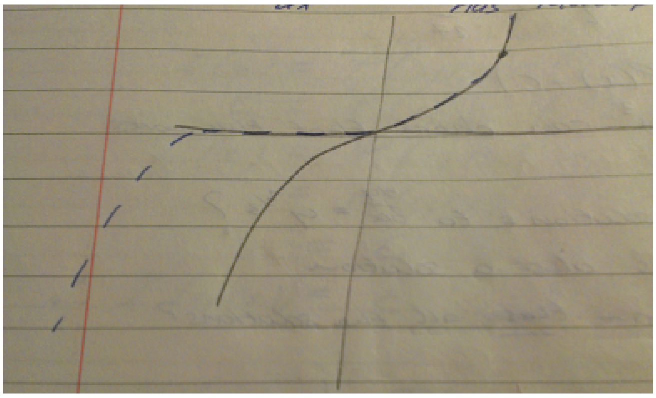
$$= \lim_{x \to -c_1} \frac{y(x) - y(-c_1)}{x + c_1} = \begin{cases} 0 & -c_2 \leq x < -c_1 \\ \frac{(x+c_1)^2}{27} & -c_1 < x \end{cases}$$

$$\lim_{x \to -c_1} \frac{y(x) - y(-c_1)}{x + c_1} = 0$$

so $y'(-c_1) = y(-c_1) = 0$ and $y'(-c_1) = y(-c_1)^{\frac{2}{3}}$ c_2 is similar so $\frac{dy}{dx} = y^{\frac{2}{3}}$ everywhere. Note the non trivial part is the existence of y' at $-c_1,\ -c_2$
Unlike the other examples, the initial value problem

$$\frac{dy}{dx} = y^{\frac{2}{3}}\ y(x_0) = y_0$$

has multiple solutions.



2.2.2. *Exact Equations.*

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$$

with some conditions on M, N
Suppose $I(x) = U(x, y(x)) = C$ Then

$$I'(x) = \frac{\partial U}{\partial x}(x, y(x)) + \frac{\partial U}{\partial y}(x, y(x))\frac{dy}{dx} = 0 \text{ if U continuously differentiable}$$

$$\frac{dy}{dx} = \frac{-\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}}$$

$$M(x,y) = \frac{\partial U}{\partial x}(x,y) \; N(x,y) = -\frac{\partial U}{\partial y}(x,y)$$

If U is twice continuously differentiable then

$$-\frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

There is no U with $\frac{\partial U}{\partial x} = M, \; \frac{\partial U}{\partial y} = -N$ unless $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$

LEMMA 2.4. *(Poincaré) If in a disc* $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0, \; M, N \in C^1$ *then there is a* $U \in C^2$ *in the same disc with centre* ξ_η *where* $\frac{\partial U}{\partial x} = M, \; \frac{\partial U}{\partial y} = -N$

PROOF.

$$U(x,y) = \int_0^1 [(x-\xi)M(tx + (1-t)\xi, ty + (1-t)\eta) - (y-\eta)N(tx + (1-t)\xi, ty + (1-t)\eta)]dt$$

Note: This is never the best way to find U. □

Example $\frac{dy}{dx} = \frac{2x-y}{x-2y}$
$M(x,y) = 2x - y \; N(x,y) = x - 2y \; \frac{\partial M}{\partial y} = -1 \; \frac{\partial N}{\partial x} = 1 \; \frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$
So $\frac{\partial U}{\partial x} = M = 2x - y \; \frac{\partial U}{\partial y} = -N = -x + 2y$

$$U(x,y) = x^2 - xy + f(y), \; f(y) \text{ a sort of constant of integration.}$$

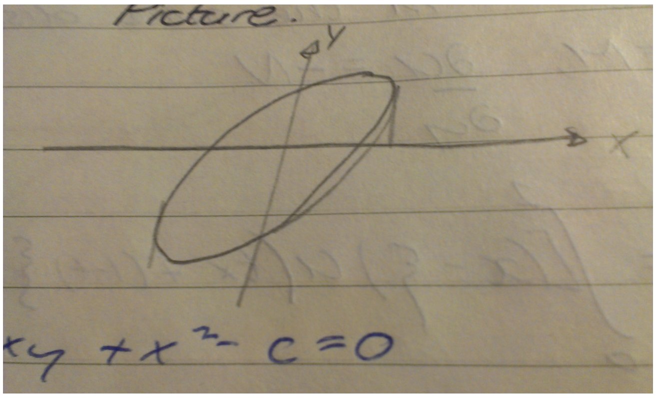
$\frac{\partial U}{\partial y} = -x - 2y = x + f'(y)$ so $f'(y) = 2y \;\Rightarrow f(y) = y^2 + C$ a constant of integration. We can take $C = 0$ so $U(x,y) = x^2 - xy + y^2$ and $\frac{dy}{dx} = \frac{2x-y}{x-2y} \iff U(x, y(x))$ is constant. $U(x, y(x)) = U(x_0, y(x_0) = U(x_0, y_0)$ so $x^2 - xy + y^2 = x_0^2 - x_0 y_0 + y_0^2 = c$ which is the equation of an ellipse.

$$y^2 - xy + x^2 - c = 0$$

From the quadratic formula,

$$y(x) = \frac{x \pm \sqrt{(-x)^2 - 4(x^2 - c)}}{2} = \frac{x \pm \sqrt{4c - 3x^2}}{2}$$

which is defined only for $3x^2 \leq 4c$, i.e, $|x| \leq \sqrt{\frac{4c}{3}} \leq \sqrt{\frac{4}{3}(x_0^2 - x_0 y_0 + y_0^2)}$ So it is differentiable for $|x| \leq \sqrt{\frac{4}{3}(x_0^2 - x_0 y_0 + y_0^2)}$



To choose the sign, use the initial conditions,

$$y_0 = \frac{x_0}{2} \pm \frac{\sqrt{4c - x_0^2}}{2}$$

Choose $+$ if $y_0 > \frac{x_0}{2}$, choose $-$ if $y_0 < \frac{x_0}{2}$

$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$ is called exact if $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0,\ \frac{dy}{dx} = F(x, y)$ It is exact if there are M, N such that $M = NF$ and $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$
Question: How do we know if there are such M, N? **Answer:** We don't

Example

$$\frac{dy}{dx} = \frac{x + y}{x - y} \Rightarrow M = x + y\ N = x - y\ \frac{\partial M}{\partial y} = 1\ \frac{\partial N}{\partial x} = 1\ 2 \neq 0$$

There is no U such that $\frac{\partial U}{\partial x} = x + y,\ \frac{\partial U}{\partial y} = -x + y$. This choice of M, N doesn't work. Does some other choice work? Yes, but hard to find. What else can go wrong?

- May not be able to evaluate integrals in closed form
- $U(x, y) = C$ may not be solvable for y as an *explicit* function of x.

For example, $\int e^{-x^2} dx$ and $\int y^{-1} e^{-y} dy$ have no elementary indefinite integrals.
Example

$$\frac{dy}{dx} = \frac{1 + \cos(x + y)}{1 - \cos(x + y)}$$

Take $M(x, y) = 1 + \cos(x + y) = \frac{\partial U}{\partial x}$ and $N(x, y) = 1 - \cos(x + y) = -\frac{\partial U}{\partial y}$ while $\frac{\partial M}{\partial y} = -\sin(x + y),\ \frac{\partial N}{\partial x} = \sin(x + y)$ so $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$ Integrating, $U(x, y) = x + \sin(x + y) + f(y)$ where $f(y)$ is a 'constant' of integration. $\frac{\partial U}{\partial y} = \cos(x + y) + f'(y) = -N(x, y) = \cos(x + y) - 1$ So $f(y) = -y + C$ where C is a constant of integration, which we can ignore as we only need one invariant. Then $U(x, y) = x - y + \sin(x + y)$
To find solutions: $x - y + \sin(x + y) = x_0 - y_0 + \sin(x_0 + y_0) = c$ We can plot $y - x = \sin(x + y) - c$ which gives different solutions for different values of c

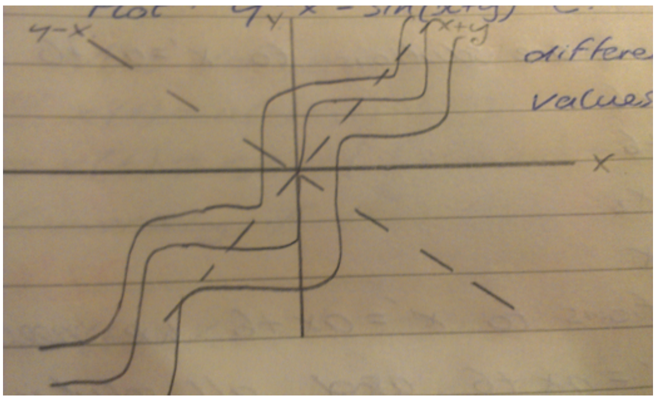


FIGURE 1. Rotated Sine waves for different C

2.3. Homogeneous First Order Linear Equations. Consider $\frac{dx}{dt} = a(t)x$ and note this is a separable equation so we can solve it.
$\frac{dx}{x} = a(t)dt$ and we integrate from (x_0, t_0)
Then, $\log(x) - \log(x_0) = \int\limits_{t_0}^{t} a(s)ds$ so $\frac{x}{x_0} = \exp(\int\limits_{t_0}^{t} a(s)ds)$ so

$$x(t) = x_0 \exp(\int\limits_{t_0}^{t} a(s)ds)$$

Example $x'(t) = tx$ as was solved previously.
Then

$$x(t) = x_0 \exp(\int\limits_{t_0}^{t} sds) = x_0 \exp(\frac{t^2}{2} - \frac{t_0^2}{2}) = x_0 \exp(\frac{-t_0^2}{2}) \exp(\frac{t^2}{2}) = c \exp(\frac{t^2}{2})$$

2.4. Inhomogeneous Linear Equations. Of the form: $\frac{dx}{dt} = a(t)x + b(t),\ x(t_0) = x_0$
Note: If x_1, x_2 are solutions to $x' = ax + b$ i.e $x_1' = ax_1 + b,\ x_2' = ax_2 + b$ then $x = x_1 - x_2$ satisfies $x' = ax$
To find all solutions to $x' = ax + b$ we need one solution to $x' = ax + b$ and all solutions to $x' = ax$
For the homogeneous problem

$$y(t) = x(t) \exp\left(-\int\limits_{t_0}^{t} a(s)ds\right)$$

is an invariant, and $x(t) = c \exp\left(\int\limits_{t_0}^{t} a(s)ds\right)$
Question: Is y an invariant of $x' = ax + b$?
Answer: No if $b \neq 0$

$$y(t) = x'(t) \exp\left(-\int\limits_{t_0}^{t} a(s)ds\right) - x(t) \exp\left(-\int\limits_{t_0}^{t} a(s)ds\right) a(t)$$

$$= [x'(t) - a(t)x(t)] \exp\left(-\int\limits_{t_0}^{t} a(s)ds\right) = b(t) \exp\left(-\int\limits_{t_0}^{t} a(s)ds\right)$$

Note: There are no $x's$ on RHS.

$$y(t) - y(t_0) = \int\limits_{t_0}^{t} y'(s)ds = \int\limits_{t_0}^{t} b(s) \exp\left(-\int\limits_{t_0}^{s} a(r)dr\right) ds$$

$$y(t) = y_0 + \int\limits_{t_0}^{t} b(s) \exp\left(-\int\limits_{t_0}^{s} a(r)dr\right) ds$$

Then, $x(t) = y(t) \exp\left(\int\limits_{t_0}^{t} a(s)ds\right)$ so, using above,

$$x(t) = y_0 \exp\left(\int\limits_{t_0}^{t} a(s)ds\right) + \exp\left(\int\limits_{t_0}^{t} a(r)dr\right) \int\limits_{t_0}^{t} \exp\left(-\int\limits_{t_0}^{t} a(r)dr\right) b(s)ds$$

Substituting in above shows $x_0 = y_0$ so

$$x(t) = x_0 \exp\left(\int\limits_{t_0}^{t} a(s)ds\right) + \int\limits_{t_0}^{t} \exp\left(\int\limits_{s}^{t} a(r)dr\right) b(s)ds$$

What did we just prove?
If

$$x'(t) = a(t)x(t) + b(t),\ x(t_0) = x_0 \tag{3}$$

then

$$x(t) = x_0 \exp\left(\int_{t_0}^{t} a(s)ds\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} a(r)dr\right) b(s)ds \tag{4}$$

Conversely, if (4), then (3).

PROOF.

$$x'(t) = x_0 \exp\left(\int_{t_0}^{t} a(s)ds\right) a(t) + \exp\left(\int_{t}^{t} a(r)dr\right) b(t) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} a(r)dr\right) a(t)b(s)ds$$

$$= a(t)x(t) + b(t) \text{ and } x(t_0) = x_0$$

□

Example $x' = tx + 2t - t^3$. Then $a(t) = t,\ b(t) = 2t - t^3$

$$x(t) = x_0 \exp\left(\int_{t_0}^{t} sds\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} rdr\right) (2s - s^3)ds$$

With initial conditions $(t_0, x_0) = (0, 0)$ then

$$x(t) = \int_{0}^{t} \exp\left(\int_{s}^{t} rdr\right) (2s - s^3)ds = \int_{0}^{t} \exp\left(\frac{t^2}{2} - \frac{s^2}{2}\right) (2s - s^3)ds$$

Changing the variable to $\sigma,\ \sigma = \frac{s^2}{2},\ d\sigma = sds$ then

$$x(t) = 2\exp\left(\frac{t^2}{2}\right) \int_{0}^{\frac{t^2}{2}} \exp(-\sigma)(1-\sigma)d\sigma$$

where

$$\int_{0}^{\frac{t^2}{2}} \exp(-\sigma)(1-\sigma)d\sigma = -\int_{0}^{\frac{t^2}{2}} (1-\sigma)\frac{d}{d\sigma}\exp(-\sigma)d\sigma$$

$$= -\int_{0}^{\frac{t^2}{2}} \frac{d}{d\sigma}[(1-\sigma)\exp(-\sigma)]d\sigma + \int_{0}^{\frac{t^2}{2}} \exp(-\sigma)\frac{d}{d\sigma}(1-\sigma)d\sigma$$

Also,

$$\int_{0}^{\frac{t^2}{2}} \exp(-\sigma)\frac{d}{d\sigma}(1-\sigma)d\sigma = \int_{0}^{\frac{t^2}{2}} (-\exp(-\sigma))d\sigma = \int_{0}^{\frac{t^2}{2}} \frac{d}{d\sigma}(\exp(-\sigma))d\sigma$$

so

$$\int_{0}^{\frac{t^2}{2}} \exp(-\sigma)(1-\sigma)d\sigma = \int_{0}^{\frac{t^2}{2}} \frac{d}{d\sigma}[\sigma\exp(-\sigma)]d\sigma = \frac{t^2}{2}\exp\left(-\frac{t^2}{2}\right)$$

so $x(t) = t^2$ satisfies $x' = tx + 2t - t^3$.

3. Linear Systems

Systems of the form:

$$x(t) = A(t)x(t) + b(t)\ x(t_0) = x_0$$

where t, t_0 is a scalar, x_0 a vector, x, b are vector valued functions and A is a (square) matrix valued function.
Note: No loss of generality in restricting to first order.
We hope that...

$$x(t) = \exp\left(\int_{t_0}^{t} A(s)ds\right) x_0 + \int_{t_0}^{t} \exp\left(\int_{s}^{t} A(r)drb(s)\right) ds$$

Questions:

(1) Can we define a matrix exponential sensibly?
(2) Is this a solution to $x' = Ax + b$?

Answers:

(1) Yes, mostly.
(2) No in general, sometimes yes.

3.1. Matrices.

DEFINITION 3.1. *For*

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix}$$

we define the matrix norm $||A||$ *to be*

$$||A|| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{i,j}|^2}$$

$||A||$ has the following properties

- $||A|| = \sqrt{trace(A\overline{A^t})} = \sqrt{trace(\overline{A^t}A)}$
- $||A+B|| \leq ||A|| + ||B||$
- $||\mu A|| = |\mu|||A||$
- $||AB|| \leq ||A||||B||$ note inequality here.

Sequences and Limits of Sequences of matrices can also be defined.

DEFINITION 3.2. *We say* $\lim_{n\to\infty} A_n = L$ *if* $\forall\epsilon > 0, \exists N$ *such that* $n > N \Rightarrow ||A_n - L|| < \epsilon$

Properties:

- $\lim_{n\to\infty}(\mu A + \nu B) = \mu \lim_{n\to\infty} A_n + \nu \lim_{n\to\infty} B_n$
- $\lim_{n\to\infty} A_n B_n = (\lim_{n\to\infty} A_n)(\lim_{n\to\infty} B_n)$

A series converges if the sequence of partial sums does.

Exponential Series: $\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$ converges for all A.

Examples:

$$\exp([a]) = \sum_{n=0}^{\infty}\frac{1}{n!}[a]^n = \sum_{n=0}^{\infty}\left(\frac{1}{n!}a^n\right) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}a^n\right) = [\exp(a)]$$

$$\exp\left(\begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{pmatrix}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{pmatrix}^n = \sum_{n=0}^{\infty}\begin{pmatrix} \frac{\lambda_1^n}{n!} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{\lambda_m^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} \exp(\lambda_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \exp(\lambda_m) \end{pmatrix}$$

Another example:

$$\exp\left(\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}\right) = \sum_{n=0}^{\infty}\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}^n$$

$$\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}^n = \begin{cases} \begin{pmatrix} \theta^n & 0 \\ 0 & \theta^n \end{pmatrix} & \text{if } n \equiv 0 (mod 4) \\ \begin{pmatrix} 0 & \theta^n \\ -\theta^n & 0 \end{pmatrix} & \text{if } n \equiv 1\ (mod 4) \\ \begin{pmatrix} -\theta^n & 0 \\ 0 & -\theta^n \end{pmatrix} & \text{if } n \equiv 2\ (mod 4) \\ \begin{pmatrix} 0 & -\theta^n \\ \theta^n & 0 \end{pmatrix} & \text{if } n \equiv 3\ (mod 4) \end{cases}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k!}\begin{pmatrix} \theta^{2k} & 0 \\ 0 & \theta^{2k} \end{pmatrix} + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\begin{pmatrix} 0 & \theta^{2k+1} \\ -\theta^{2k+1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k!}\theta^{2k} & 0 \\ 0 & \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k!}\theta^{2k} \end{pmatrix} + \begin{pmatrix} 0 & \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\theta^{2k+1} \\ -\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\theta^{2k+1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Example

$$\exp\left(\begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix} + 0 + 0 = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}$$

as all higher powers of the matrix are zero. It then follows that

$$\exp\left(\begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}\right)\exp\left(\begin{pmatrix} 0 & 0 \\ -\theta & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} = \begin{pmatrix} 1-\theta^2 & \theta \\ -\theta & 1 \end{pmatrix}$$

but a similar computation yields

$$\exp\left(\begin{pmatrix} 0 & 0 \\ -\theta & 0 \end{pmatrix}\right)\exp\left(\begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & \theta \\ -\theta & 1-\theta^2 \end{pmatrix}$$

and also

$$\exp\left(\begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\theta & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

In general, $\exp(A)\exp(B)$ and $\exp(B)\exp(A)$ are distinct. $(A+B)^n = \sum\limits_{j+k=n} \frac{n!}{j!k!}A^jB^k$ if $AB = BA$, so $\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A)$ if $BA = AB$

Example: $\exp((s+t)A) = \exp(sA+tA) = \exp(sA)\exp(tA) = \exp(tA)\exp(sA)$ if $s = -t$.
$I = \exp(-tA)\exp(tA) = \exp(tA)\exp(-tA)$ and

$$\frac{\exp((s+t)A) - \exp(tA)}{(s+t)-t} = \frac{\exp(sA)\exp(tA) - \exp(tA)}{s} = \frac{(\exp(sA)-I)(\exp(tA))}{s}$$

$$= \left(\sum_{n=1}^{\infty} \frac{s^{n-1}}{n!}A^n\right)\exp(tA), \text{ and} \sum_{n=1}^{\infty} \frac{s^{n-1}}{n!}A^n = A$$

so

$$\frac{d}{dt}\exp(tA) = \lim_{s\to 0} \frac{\exp((s+t)A) - \exp(tA)}{s} = A\exp(tA) = \exp(tA)A$$

For example, $\frac{d}{dt}[\exp(tA)x_0] = [\frac{d}{dt}\exp(tA)]x_0 + 0 = A\exp(tA)x_0$ i.e, $x(t) = \exp(tA)x_0$ is a solution of $x' = Ax$.
Conversely, if $x' = Ax$ then set $y = \exp(-tA)x$ so $\frac{dy}{dx} = -\exp(-tA)Ax + \exp(-tA)x'$.
$y' = \exp(-tA)(x' - Ax) = 0 \Rightarrow y$ is constant. $x_0 = \exp(-tA)x(t) \Rightarrow \exp(tA)x_0 = \exp(tA)\exp(-tA)x(t) = x(t)$
So $x(t) = \exp(tA)x_0$ is the unique solution to $x' = Ax,\ x(0) = x_0$ and $x(t) = \exp((t-t_0)A)x_0$ is the solution of $x'(t) = Ax(t),\ x(t_0) = x_0$

Example $x'' + x = 0,\ x' = v,\ v' = -x$ which can be represented as vectors and matrices. For example

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \exp\left((t-t_0)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\begin{pmatrix} x(t_0) \\ v(t_0) \end{pmatrix} = \begin{pmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{pmatrix}\begin{pmatrix} x(t_0) \\ v(t_0) \end{pmatrix}$$

$$\Rightarrow x(t) = x(t_0)\cos(t-t_0) + x'(t_0)\sin(t-t_0)$$

This works only because A is constant!

For $x'(t) = A(t)x(t),\ x(t_0) = x_0$ in general, $\exp((t-t_0)A(t))$ is not a solution, nor is $\exp\left(\int\limits_{t_0}^{t} A(s)ds\right)$

Example

$$A(t) = \begin{pmatrix} 0 & \frac{-1}{t^2} \\ 1 & 0 \end{pmatrix} \Rightarrow \exp\left(t\begin{pmatrix} 0 & \frac{-1}{t^2} \\ 1 & 0 \end{pmatrix}\right) = \exp\left(\begin{pmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{pmatrix}\right)$$

$$\begin{pmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{pmatrix}^n = \begin{cases} (-1)^k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } n = 2k \\ (-1)^{2k+1}\begin{pmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{pmatrix} & \text{if } n = 2k+1 \end{cases}$$

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\begin{pmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{pmatrix}$$

$$= \cos(1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin(1)\begin{pmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{pmatrix}$$

Note that

$$x = \begin{pmatrix} \cos(1) & \frac{-\sin(1)}{t} \\ t\sin(1) & \cos(1) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

is not a solution of $x'(t) = A(t)x(t)$ for some $\begin{pmatrix} a \\ b \end{pmatrix}$

$$x'(t) = \begin{pmatrix} 0 & \frac{\sin(1)}{t^2} \\ \sin(1) & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$Ax(t) = \begin{pmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{pmatrix}\begin{pmatrix} \cos(1) & \frac{-\sin(1)}{t} \\ t\sin(1) & \cos(1) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\sin(1) & \frac{-\cos(1)}{t} \\ \cos(1) & \frac{-\sin(1)}{t} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

So for $x' = Ax$ we have the general solution $x(t) = \exp\left((t-t_0)A\right)x(t_0)$ where $\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n$. We aim to solve this more explicitly.

Special Case: Suppose A is diagonalizable, i.e, $Av_1 = v_1\lambda_1, \ldots, Av_m = v_m\lambda_m$ where A is an $m \times m$ matrix, λ is a scalar, a 1×1 matrix, and v a $m \times 1$ matrix. Also, $v_1, \ldots v_m$ are eigenvalues and $\lambda_1, \ldots \lambda_m$ are eigenvalues. Then $AV = V\Lambda$, where

$$V = \begin{pmatrix} \vdots & & \vdots \\ v_1 & \ldots & v_m \\ \vdots & & \vdots \end{pmatrix}, \ \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_m \end{pmatrix}$$

Then $\{v_1, \ldots, v_m\}$ are a basis, so V is invertible.

$$AV = V\Lambda \ \Rightarrow \ A = V\Lambda v^{-1} \ \Rightarrow \ \exp(tA) = V\exp(t\Lambda^n)V^{-1}$$

Example Take $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which has eigenvalues $\lambda_1 = 1$ for $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -1$ for $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so

$$A\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $A = V\Lambda v^{-1}$ and $v^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ so $\exp(tA) = V\exp(t\Lambda)V^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\exp\left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}\right)\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^t+e^{-t}}{2} & \frac{e^t-e^{-t}}{2} \\ \frac{e^t-e^{-t}}{2} & \frac{e^t+e^{-t}}{2} \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

So,

$$x' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}x \ \Rightarrow \ x(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}x_0$$

What if A is not diagonalizable?

THEOREM 3.3 (Jordan Decomposition). *If A is a complex square matrix then $A = VJV^{-1}$ where $J = D+N$ for matrices D, N and:*

- *D is diagonal*
- *N is nilpotent (some power of it is zero)*
- *$DN = ND$*

So D and N are of the form:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_m \end{pmatrix}, \ N = \begin{pmatrix} 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ 0 & \ldots & \ddots & \\ \vdots & & & \end{pmatrix}$$ *for say the first 3 eigenvalues equal*

Then $\exp(tA) = V\exp(tJ)V^{-1} = V\exp(tD + tN)V^{-1} = V\exp(tD)\exp(tN)V^{-1}$.

Example Consider $y'' - 3y' + 2y = 0$. We can use reduction of order as before, giving

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

where the eigenvalues are the roots of the characteristic polynomial, $\det(\lambda I - A) = \lambda^3 - 3\lambda + 2 = 0 = (\lambda + 2)(\lambda - 1)^2$, so the eigenvalues are $\lambda = -2, 1, 1$ and the eigenvectors can then be found

$$A\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}(-2)\,,\; A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1)\,,\; A\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}(1) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

so then, we have

$$A\begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where $V = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and J can be decomposed into D and N, where

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\; N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ which implies}$$

$$\exp(tD) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix},\; \exp(tN) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \exp(tJ) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}$$

$$\exp(tA) = V\begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}V^{-1} \text{ so } x(t) = \exp(tA)x(0)$$

$$\begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix} = \exp(tA)\begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \end{pmatrix} = V\begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}V^{-1}\begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \end{pmatrix}$$

and $V^{-1}\begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \end{pmatrix}$ is constant so we can solve explicitly to get

$y(t) = c_1e^{-2t} + c_2e^t + c_3te^t$ and the coefficients can be determined from the initial condition and differentiating this general solution.

- $y(0)c_1 + c_2$
- $y'(0) = -2c_1 + c_2c + c_3$
- $y''(0) = 4c_1 + c2 + x^{(m-1)}(0)c_4$

and then solve for the coefficients.

3.2. Linear Constant Coefficient Homogeneous Equations, Scalar.

Consider the equation

$$c_mx^{(m)} + c_{m-1}x^{(m-1)} + \ldots + c_0x = 0$$

We can carry out reduction of order as before to give us the following system:

$$\frac{d}{dt}\begin{pmatrix} x & \\ x^1 & x^{(m-1)}(0) \\ \vdots & \\ x^{(m-1)} & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & 0 & \ldots \\ 0 & \ldots & & \ddots & \\ & & & & 1 \\ \frac{-c_0}{c_m} & \ldots & & & \frac{-c_{m-1}}{c_m} \end{pmatrix}\begin{pmatrix} x \\ x^1 \\ \vdots \\ x^{(m-1)} \end{pmatrix} \text{ i.e } \frac{dx}{dt} = Ax$$

We now need the characteristic polynomial of this system, $f(\lambda)$:

$$P(\lambda) = \det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 & 0 & \ldots & 0 \\ \vdots & \lambda & -1 & 0 & \ldots \\ 0 & \ldots & & \ddots & \\ & & & \lambda & -1 \\ \frac{c_0}{c_m} & \ldots & & & \lambda + \frac{c_{m-1}}{c_m} \end{pmatrix}$$

$$= \frac{c_{m-1}}{c_m}\lambda^{m-1} + \ldots + \frac{c_0}{c_m}\lambda^0 = \frac{c_{m-1}\lambda^{m-1} + \ldots + c_1\lambda + c_0}{c_m}$$

and $A = VJV^{-1}$ where $J = D + N$ as before. So the roots are $\lambda_1, \ldots, \lambda_m$ and

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \\ 0 & & \lambda_m \end{pmatrix}, N = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & 0 & \ldots \\ 0 & \ldots & & \ddots & \\ & & & & \\ & \ldots & & & \end{pmatrix}$$

with the 1's above the diagonal (Jordan Blocks) and 0 along the diagonal. This is because we require $DN = ND$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & 0 & \ldots \\ 0 & \ldots & & \ddots & \\ & & & & 1 \\ \frac{-c_0}{c_m} & \ldots & & & \frac{-c_{m-1}}{c_m} \end{pmatrix},\ A^2 = \begin{pmatrix} 0 & 0 & 1 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 & \ldots \\ 0 & \ldots & 0 & & 1 \\ & stuff & & & \\ stuff & \ldots & & & \end{pmatrix}$$

$$A^{m-1} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ stuff & \ldots & & \ldots & \\ \vdots & \ldots & & \ddots & \\ & \ldots & & & \\ \ldots & & & & \end{pmatrix},\ A^0 = I$$

and also $I, A^2, \ldots, A^{m-1}$ are necessarily linearly independent. $P(A) = 0$ and P is non zero so the degree of $P \geq m$. So the minimal polynomial is equal to the characteristic polynomial. This means N has 1's wherever it can (along the super-diagonal). Then we have

$$\exp(tD) = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ 0 & \ddots & \\ 0 & & e^{\lambda_m t} \end{pmatrix},\ \exp(tN) = \begin{pmatrix} 1 & t & \frac{t^2}{2} & 0 & \ldots & & & 0 \\ 0 & 1 & t & 0 & \ldots & & & 0 \\ 0 & 0 & 1 & 0 & \ldots & & & 0 \\ 0 & \ldots & & 1 & t & 0 & \ldots & 0 \\ 0 & \ldots & & & 1 & 0 & \ldots & \\ 0 & \ldots & & & 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & \ldots & & & 0 & 1 & t & \frac{t^2}{2} \\ \ldots & & & & 0 & 0 & 0 & 1 \end{pmatrix}$$

for each of the corresponding $k \times k$ blocks. Therefore,

$$\exp(tJ) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & 0 & \ldots \\ 0 & \ddots & & \\ \vdots & & e^{\lambda t} & \\ & & & \end{pmatrix}$$

for each of the $k \times k$ blocks. So

$$x(t) = \begin{pmatrix} x(t) \\ x^1(t) \\ \vdots \\ x^{(m-1)}(t) \end{pmatrix} = \exp(tA) \begin{pmatrix} x(0) \\ x^1(0) \\ \vdots \\ x^{(m-1)}(0) \end{pmatrix} = V \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & 0 & \ldots \\ 0 & \ddots & & \\ \vdots & & e^{\lambda t} & \\ & & & \end{pmatrix} V^{-1} \begin{pmatrix} x(0) \\ x^1(0) \\ \vdots \\ x^{(m-1)}(0) \end{pmatrix}$$

So the solution is some linear combination of $t^j e^{\lambda t}$ where λ is a root of the characteristic polynomial and j is an integer, where j is less than or equal to the multiplicity of P. The set of all solutions is a vector space with basis $\{t^j e^{\lambda t}\}$.

To solve this as an initial value problem, with, say,

- $x(0) = \ldots$
- $x'(0) = \ldots$
- $\vdots$
- $x^{(m-1)}(0) = \ldots$

we use the *Method of Undetermined Coefficients*:

- $x(t) = \alpha_1 e^{\lambda t} + \alpha_2 t e^{\lambda t} \ldots + \alpha_m \ldots$ in terms of each of the basis vectors.
- $\vdots$
- $x^{(m-1)}(t) = \alpha_1 \lambda^{m-1} e^{\lambda t} + \ldots$ and solve the system of linear equations.

Example Forced Damped Harmonic Oscillator.

(5) $$ax'' + bx' + cx = \cos(\Omega t)$$

which is an inhomogeneous equation. To solve this we use a trick. Differentiating twice we get

(6) $$ax'''' + bx''' + cx'' = -\Omega^2 \cos(\Omega t)$$

and then $(6) + \Omega^2(5)$ gives

(7) $$= ax'''' + bx''' + (c + a\Omega^2)x'' + b\Omega^2 x' + c\Omega^2 x = 0$$

which is an homogeneous system. It should be noted that (5) imples (7) but the converse is not true. The characteristic polynomial of this 4×4 system is

$$P(\lambda) = a\lambda^4 + b\lambda^3 + (c + \Omega^2)\lambda^2 + b\Omega^2\lambda + c\Omega^2 = (a\lambda^2 + b\lambda + c)(\lambda^2 + \Omega^2)$$

which has roots $\lambda = \pm i\Omega$ and $\lambda = \frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$. We then have 3 cases to consider:

1. $\Delta > 0$, the *Overdamped Harmonic Oscillator*
2. $\Delta = 0$, the *Critically Damped Harmonic Oscillator*
3. $\Delta < 0$, the *Underdamped Harmonic Oscillator*

The last of these cases is the most interesting so we will focus on this one.
Set $\omega = \frac{\sqrt{\Delta}}{2a}$ and $r = \frac{b}{2a}$ so that the roots of the polynomial are $\pm i\Omega,\ r \pm \omega$ and a basis of the solutions is $\{e^{i\Omega t}, e^{-i\Omega t}, e^{-rt+i\omega t}, e^{-rt-i\omega t}\}$. This would not be a basis if $\Omega = 0$ or $\omega = 0$ but we exclude these cases as then system would then be critically damped and homogeneous, respectively. The only other time when this will not be a basis is if $r = 0$ and $\omega = \Omega$, which is known as *resonance*. The basis we have is equivalent to $\{\cos(\Omega t), \sin(\Omega t), e^{-rt}\cos(\omega t), e^{-rt}\sin(\omega t)\}$ so we get

$$x(t) = \alpha\cos(\Omega t) + \beta\sin(\Omega t) + \gamma e^{-rt}\cos(\omega t) + \delta e^{-rt}\sin(\omega t)$$

for constants $\alpha, \beta, \gamma, \delta$ differentiating this we get

$$x'(t) = \beta\Omega\cos(\Omega t) - \alpha\Omega\sin(\Omega t) + (\delta\omega - r\gamma)e^{-rt}\cos(\omega t) - (\gamma\omega + r\delta)e^{-rt}\sin(\omega t)$$

and similarly for $x''(t)$ and $x'''(t)$, as required. Evaluating these at 0 we then have a system of linear equations

- $x(0) = \alpha + \gamma$
- $x'(0) = \beta\Omega + \delta\omega - \gamma r$
- $x''(0) = \ldots$
- $x'''(0) = \ldots$

which can be solved for $\alpha, \beta, \gamma, \delta$ in terms of $x(0), x'(0), x''(0), x'''(0)$. So we have:

$$ax'' + bx' + cx = \cos(\Omega t)$$
$$ax''' + bx'' + cx' = \Omega\cos(\Omega t)$$

which give

$$x''(0) = \frac{-b}{a}x'(0) - \frac{c}{a}x(0) + \frac{1}{a}$$
$$x'''(0) = \frac{-b}{a}x''(0) - \frac{c}{a}x'(0)$$

which give

$$x(t) = \frac{(r^2 + \omega^2 - \Omega^2)(\cos(\Omega t) - e^{-rt}\cos(\omega t) - \frac{r}{\omega}e^{-rt}\sin(\omega t)) + 2r\Omega(\sin(\omega t) - \frac{\Omega}{\omega}e^{-rt}\sin(\omega t))}{r^4 + 2r^2\omega^2 + 2r^2\Omega^2 + \omega^4 - 2\omega^2\Omega^2 + \Omega^4}$$
$$+x(o)\left(e^{-rt}\cos(\omega t) + \frac{r}{\omega}e^{-rt}\sin(\omega t)\right) + x'(0)\frac{1}{\omega}e^{-rt}\sin(\omega t)$$

If $r = 0$ and $\omega = \Omega$ the denominator vanishes and so $\{\cos(\Omega t), \sin(\Omega t), e^{-rt}\cos(\omega t), e^{-rt}\sin(\omega t)\}$ is no longer a basis for solutions to (7). To find a basis in this case we have the characteristic polynomial $(\lambda^2 + \Omega^2)(\lambda^2 - 2r\lambda + r^2 + \omega^2) = (\lambda^2 + \omega^2)$ with resonance. Then, the basis should be

$$\{e^{i\Omega t}, te^{i\Omega t}, e^{-i\Omega t}, x^{(m-1)}(0)te^{-i\Omega t}\} = \{\cos(\Omega t), \sin(\Omega t), t\cos(\Omega t), t\sin(\Omega t)\}$$

Then, similarly to above we can find a general solution and solve for the constant coefficients in terms of $x(0),\ x'(0),\ x''(0),\ x'''(0)$ and write $x''(0),\ x'''(0)$ in terms of $x'(0),\ x(0)$.

This method is known as the *method of undetermined coefficients*, which works for:

- Linear constant coefficient homogeneous scalar equations.

- Some linear constant coefficient inhomogeneous scalar equations. It works for those whose RHS satisfy a linear constant coefficient homogeneous equation.

This means, for polynomials p and q we have

$$p\left(\frac{d}{dt}\right)x = f \text{ where } q\left(\frac{d}{dt}\right)f = 0$$

then

(8)
$$q\left(\frac{d}{dt}\right)p\left(\frac{d}{dt}\right)x = 0$$

which is the characteristic polynomial. In the previous example, $p(\lambda) = \lambda^2 - 2r\lambda + r^2 + \omega^2$ and $q(\lambda) = \lambda^2 + \Omega^2$ and

$$\left(\frac{d}{dt^2} + \Omega^2\right)\cos(\Omega t) = 0$$

A basis for (8) is of the form $t^j e^{\lambda t}$ where λ is a root of pq and j <multiplicity of pq

3.2.1. *Solutions using Fundamental Matrices. What about a more general function f?*
i.e, not ones satisfying $q\left(\frac{d}{dt}\right)f = 0$ for any q
By reduction of order, we can write it as a first order system $x'(t) = Ax(t) + b(t)$
More generally, we can consider

$$x'(t) = A(t)x(t) + b(t)$$

possibly after reduction of order. We then need the following definition.

DEFINITION 3.4. $W(s,t)$ *is said to be a fundamental matrix if*
- $\frac{\partial}{\partial t}W(s,t) = A(t)W(t)$
- $W(t,t) = I$

Example If A is constant, $W(s,t) = \exp((s-t)A)$.
For general A, there is a unique fundamental matrix. Take $W(s,t)W(s,t)^{-1} = I$. Then, differentiating gives:

$$\frac{\partial}{\partial t}W(s,t)W(s,t)^{-1} + W(s,t)\frac{\partial}{\partial t}\left(W(s,t)^{-1}\right) = 0$$

and on multiplying by $W(s,t)^{-1}$ we get

$$W(s,t)^{-1}\frac{\partial}{\partial t}W(s,t)W(s,t)^{-1} + \frac{\partial}{\partial t}\left(W(s,t)^{-1}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(W^{-1}\right) = -W^{-1}\left(\frac{\partial}{\partial t}W\right)W^{-1} = -W(s,t)^{-1}A(t)W(s,t)W(s,t)^{-1} = -W(s,t)^{-1}A(t)$$

we then have the following claim.
Claim: $W(s,t)W(r,s) = W(r,t)$

PROOF. $W(r,t)^{-1}W(s,t)W(r,s) = I$ if $s = t$. Then, differentiating,

$$\frac{\partial}{\partial t}\left(W(r,t)^{-1}W(s,t)W(r,s)\right) = \left(\frac{\partial}{\partial t}W(r,t)^{-1}\right)W(s,t)W(r,s) + W(r,t)^{-1}\left(\frac{\partial}{\partial t}W(s,t)\right)W(r,s)$$
$$= -W(r,t)A(t)W(s,t)W(r,s) + W(r,t)^{-1}A(t)W(s,t)W(r,s) = 0$$

so we have $W(r,t)^{-1}W(s,t)W(r,s) = I$ for all t so $W(s,t)W(r,s) = W(r,t)$. □

Suppose $x'(t) = A(t)x(t) + b(t)$. Then, define

$$y(t) = W(t_0,t)^{-1}x(t)$$

so we have

$$y'(t) = -W(t_0,t)^{-1}A(t)x(t) + W(t_0,t)^{-1}x'(t)$$
$$= W(t_0,t)^{-1}\left(x'(t) - A(t)x(t)\right) = W(t_0,t)^{-1}b(t)$$

then we can get

$$y(t) = y(t_0) + \int_{t_0}^{t} W(t_0,s)^{-1}b(s)ds$$

and $x(t) = W(t_0,t)y(t)$

$$= W(t_0,t)y(t_0) + W(t_0,t)\int_{t_0}^{t} W(t_0,s)^{-1}b(s)ds = W(t_0,t)x(t_0) + \int_{t_0}^{t} W(s,t)b(s)ds$$

Conversely, if we have

$$x(t) = W(t_0,t)x_0 + \int_{t_0}^{t} W(s,t)b(s)ds$$

then $x'(t) = A(t)x(t) + b(t)$ and $x(t_0) = x_0$ so

$$x'(t) = A(t)W(t_0,t)x_0 + W(t,t)b(t) + \int_{t_0}^{t} A(t)W(s,t)b(s)ds$$

$$= A(t)\left(W(t_0,t)x_0 + \int_{t_0}^{t} W(s,t)b(s)ds\right) + Ib(t) = A(t)x(t) + b(t)$$

How do we find W? $\frac{d}{dt}V(t) = A(t)V(t)$ if and only if the columns of V satisfy $\frac{d}{dt}v(t) = A(t)v(t)$.
If V is invertible then $W(s,t) = V(t)V(s)^{-1}$ works and $\frac{\partial}{\partial t}W(s,t) = A(t)V(t)V(s)^{-1} = A(t)W(s,t)$ and $W(t,t) = V(t)V(t)^{-1} = I$. Then $x'(t) = A(t)x(t) + b(t)$, $x(t_0) = x_0$ gives $x(t) = W(t_0,t)x_0 + \int_{t_0}^{t} W(s,t)b(s)ds$ and $W(s,t) = V(t)V(s)^{-1}$ so the columns of V are linearly independent solutions of $v'(t) = A(t)v(t)$.

Example Consider $(1-t^2)x''(t) - 2tx'(t) = f(t)$. Let $y(t) = x'(t)$ so $y'(t) = \frac{2t}{1-t^2}y(t) + \frac{f(t)}{1-t^2}$ which can be written in matrix form, giving

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & \frac{2t}{1-t^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{f(t)}{1-t^2} \end{pmatrix}$$

which has solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = W(0,t)\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \int_0^t W(s,t)\begin{pmatrix} \\ \frac{f(s)}{1-s^2} \end{pmatrix} ds$$

to find W, solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & \frac{2t}{1-t^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

giving $x'(t) = y(t)$ and $y'(t) = \frac{2t}{1-t^2}y(t)$ so then solving for $x(t), y(t)$

$$y(t) = y(0)\exp\left(\int_0^t \frac{2s}{1-s^2}\right)ds = y(0)\exp\left(-\log(1-t^2)\right) = \frac{y(0)}{1-t^2}$$

$$x(t) = x(0) + \int_0^t x'(s)ds = x(0) + \int_0^t y(s)ds = x(0) + \int_0^t \frac{y(0)}{1-s^2}ds$$

$$= x(0) + y(o)\int_0^t \frac{1}{2}\left(\frac{1}{1-s} + \frac{1}{1+s}\right)ds = x(0) + \frac{y(0)}{2}\log\left(\frac{1+t}{1-t}\right)$$

and we know $y(t) = \frac{y(0)}{2}$ from before which gives solutions

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \\ \frac{1}{1-t^2} \end{pmatrix}$$

so

$$V(t) = \begin{pmatrix} 1 & \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) \\ 0 & \frac{t}{1-t^2} \end{pmatrix}$$

$$V(s)^{-1} = \begin{pmatrix} 1 & -\frac{1}{2}(1+s^2)\log\left(\frac{1+s}{1-s}\right) \\ 0 & 1+s^2 \end{pmatrix}$$

and $W(s,t) = V(t)V(s)^{-1}$ which gives

$$W(s,t) = \begin{pmatrix} 1 & \frac{1}{2}(1+s^2)\log\left(\frac{1+t}{1-t}\right) - \frac{1}{2}(1+t^2)\log\left(\frac{1+s}{1-s}\right) \\ 0 & \frac{1+s^2}{1-t^2} \end{pmatrix}$$

so for the original equation $(1-t^2)x''(t) - 2tx'(t) = f(t)$ the solution is

$$x(t) = x(0) + y(0)\frac{1}{2}\left((1-s^2)\log\left(\frac{1+t}{1-t}\right) - (1-t^2)\log\left(\frac{1+s}{1-s}\right)\right)$$

$$+\frac{1}{2}\int_0^t \left((1-s^2)\log\left(\frac{1+t}{1-t}\right)-(1-t^2)\log\left(\frac{1+s}{1-s}\right)\right)\frac{f(s)}{1-s^2}ds$$

We have $\frac{\partial W}{\partial t}=A(t)W(t)$ which can also be written as

$$W'_{i,k}=\sum_{j=1}^{n}a_{i,j}(t)W_{j,k}(t)$$

and the rows of W satisfy

$$r'_i=\sum_{j=1}^{k}a_{i,j}(t)r_j(t)$$

$$W(s,t)=\begin{pmatrix} - & r_1 & - \\ - & \vdots & - \\ - & r_n & - \end{pmatrix}$$

and the determinant is linear in each row.

$$\det(W(s,t))'=\det\begin{pmatrix} - & r'_1 & - \\ - & \vdots & - \\ - & r_n & - \end{pmatrix}+\ldots+\det\begin{pmatrix} - & r_1 & - \\ - & \vdots & - \\ - & r'_n & - \end{pmatrix}$$

and also

$$det\begin{pmatrix} - & r_1 & - \\ - & \vdots & - \\ - & r'_i & - \\ - & \vdots & - \\ - & r_n & - \end{pmatrix}=\begin{pmatrix} - & r_1 & - \\ - & \vdots & - \\ - & \sum_j a_{ij}r_j & - \\ - & \vdots & - \\ - & r_n & - \end{pmatrix}=\sum_{j=1}^{n}a_{ij}det\begin{pmatrix} - & r_1 & - \\ - & \vdots & - \\ - & r_j & - \\ - & \vdots & - \\ - & r_n & - \end{pmatrix}$$

with the r_j in the ith row. We then have to consider two cases. If $i=j$ the the value is $a_{i,j}\det(W(s,t)),\ 0$ otherwise. So

$$\det(W(s,t))'=\sum_{i=1}^{n}a_{i,i}(t)\det(W(s,t))=tr(A(t))\det(W(s,t))$$

so $\det(W(s,t))$ satisfies a first order <u>scalar</u> equation.

$$\det(W(s,t))=\det(W(s,s))\exp\left(\int_s^t tr(A(r))\right)dr=\exp\left(\int_s^t tr(A(r))dr\right)$$

Example Consider $A=\begin{pmatrix} 0 & 1 \\ 0 & \frac{2t}{1-t^2}\end{pmatrix}$ then we get

$$\det(W(s,t))=\exp\left(\int_s^t\frac{2r}{1-r^2}dr\right)=\exp\left(\log\left(\frac{1-s^2}{1-t^2}\right)\right)=\frac{1-s^2}{1-t^2}$$

then

$$\det(W(s,t))=\exp\left(\int_s^t\frac{2r}{1-r^2}dr\right)=\exp\left(\log\left(\frac{1-s^2}{1-t^2}\right)\right)=\frac{1-s^2}{1-t^2}$$

and also $\det(W(s,t))=\begin{vmatrix} 1 & stuff \\ 0 & \frac{1-s^2}{1-t^2}\end{vmatrix}=\frac{1-s^2}{1-t^2}$ as it should be.

Then $\frac{d}{dt}\det(W(s,t))=tr(A(t))\det(W(s,t))$ which implies $\det(W(s,t))=\exp\left(\int_s^t tr(A(r))dr\right)$ i.e $\frac{\partial W}{\partial t}=AW$. If V satisfies $V'=AV$ then $\det(V)'=tr(A)\det(V)$ and $\det(V(t))=\det(V(S))\exp\left(\int_s^t tr(A(r))dr\right)$

3.3. Higher Order Scalar Equations. We can then consider higher order scalar equations of the form

$$c_n(t)x^{(n)}(t) + \ldots + c_1(t)x'(t) + c_0(t)x(t) = 0$$

which can be re-arranged using reduction of order to give

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ x'(t) \\ \vdots x^{(n-1)}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \vdots & & \\ 0 & \ldots & & & \\ \frac{-c_1}{c_n} & \ldots & & & \frac{-c_{n-1}}{c_n} \end{pmatrix}\begin{pmatrix} x(t) \\ x'(t) \\ \vdots x^{(n-1)}(t) \end{pmatrix}$$

calling the matrix $A(t)$.
So then we have $v'(t) = A(t)v(t)$ and $tr(A(t)) = \frac{-c_{n-1}(t)}{c_n(t)}$ so

$$\det(W(s,t)) = \exp\left(-\int_s^t \frac{c_{n-1}(r)}{c_n(r)}dr\right)$$

so it only depends on the first two coefficients.

3.3.1. *Second Order Scalar.* Focus on equations of the form

$$c_2(t)x''(t) + c_1(t)x'(t) + c_0(t)x(t)$$

then we get

$$V(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix}$$

so $\det(V(t)) = x_1(t)x_2'(t) - x_2(t)x_1'(t)$ and

$$\det(V(t)) = x_1(t)x_2'(t) - x_2(t)x_1'(t) = \det(V(s))\exp\left(\int_s^t tr(A(r))dr\right) = \det(V(s))\exp\left(-\int_s^t \frac{c_1(r)}{c_n(r)}dr\right)$$

which is a first order linear inhomogeneous scalar equation for x_2 if x_1 is given.

Example Take $(1-t^2)x'' - 2tx' = 0$. Then $x_1(t) = 1$ is a solution and

$$x_1(t)x_2'(t) - x_1'(t)x_2(t) = c\exp\left(\int_s^t \frac{2r}{1-r^2}dr\right) = c\exp\left(-\log(1-r^2)\right)\Big|_{r=s}^{r=t} = c\frac{1-s^2}{1-t^2}$$

Then we can choose $s = 0$ and $c = 1$ so $x_2'(t) = \frac{1}{1-t^2}$ which means

$$x_2(t) = x_2(s) + \int_s^t \frac{1}{1-r^2}dr$$

we chose $s = 0$ so $x_2(s) = 0$ giving

$$x_2(t) = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$$

Example

$$(1-t^2)x''(t) - 2tx'(t) + 6x(t) = 0$$

A particular solution is $x_1(t) = 3t^2 - 1$ so $x_1'(t) = 6t,\ x_1''(t) = 6$ which gives $(1-t^2)6 - 2t(6t) + 6(3t^2-1) = 0$ and we can get

$$x_1(t)x_2'(t) - x_1'(t)x_2(t) = c\frac{1}{1-t^2}$$

so $\det(V(t))$ only depends on the first two coefficients. We can choose $c = 1$ (any non zero choice will work) and $s = 0$ so $x_2(s) = 0$ (almost any choice of s will work (zeroes of the particular solution)). Then we get

$$x_2'(t) = \frac{6t}{3t^2-1}x_2(t) + \frac{1}{1-t^2}\frac{1}{3t^2-1}$$

so

$$x_2(t) = \int_0^t \exp\left(\int_s^t \frac{6r}{3r^2-1}dr\right)\frac{1}{1-s^2}\frac{1}{3s^2-1}ds$$

then $\exp\left(\int\limits_s^t \frac{6r}{3r^2-1}dr\right) - \exp\left(\log\left(\frac{3t^2-1}{3s^2-1}\right)\right) = \frac{3t^2-1}{3s^2-1}$ so

$$x_2(t) = \int\limits_0^t \frac{3t^2-1}{(3s^2-1)^2}\frac{1}{1-s^2}ds = (3t^2-1)\int\limits_0^t \frac{ds}{(1-s^2)(1-3s^2)^2}$$

We can solve $\int\limits_0^t \frac{ds}{(1-s^2)(1-3s^2)^2}$ using partial fractions:

$$\frac{1}{(1-s^2)(1-3s^2)^2} = \frac{1}{4}\frac{1}{1-s^2} + \frac{3}{4}\frac{1+3s^2}{(1-3s^2)^2} = \frac{1}{8}\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right) + \frac{3}{4}\frac{d}{ds}\left(\frac{s}{1-3s^2}\right)$$

so $\int\limits_0^t \frac{ds}{(1-s^2)(1-3s^2)^2} = \frac{1}{8}\log\left(\frac{1+t}{1-t}\right) + \frac{3}{4}\left(\frac{t}{1-3t^2}\right)$ giving

$$x_2(t) = \frac{3t^2-1}{8}\log\left(\frac{1+t}{1-t}\right) - \frac{3}{4}t$$

This method is known as *Wronskian Reduction of Order*

3.4. Method of Successive Approximations-Picard Iterations. Suppose we have $x'(t) = ax(t)$ and $x(t_0) = \xi$. Then by the fundamental theorem of Calculus we have

$$x(t) = x(t_0) + \int\limits_{t_0}^t x'(s)ds = \xi + \int\limits_{t_0}^t Ax(s)ds$$

Picard iterations are similar to Newton's method for approximating roots and take the following form

- Take an initial guess $x_{-1}(t)$
- Refine this initial guess using $x_n(t) = \xi + \int\limits_{t_0}^t Ax_{n-1}(s)ds$

Questions:

(1) How to choose x_{-1}?
(2) Does the sequence $x_n(t)$ converge?
(3) Does $x(t) = \lim\limits_{n\to\infty} x_n(t)$ satisfy $x'(t) = Ax(t),\ x(t_0) = \xi$?

Answers:

(1) It doesn't matter, so may as well take $x_{-1}(t) = 0$
(2) Yes
(3) Yes

So we have $x_{-1}(t) = 0$ and using the iteration gives $x_{-1}(t) = 0$ so

$$x_0(t) = \xi + \int\limits_{t_0}^t Ax_{-1}(s)ds = \xi$$

$$x_1(t) = \xi + \int\limits_{t_0}^t Ax_0(s)ds = \xi + \int\limits_{t_0}^t A\xi ds = \xi + (t-t_0)A\xi = (I + (t-t_0)A\xi)$$

$$x_2(t) = \xi + \int\limits_{t_0}^t Ax_1(s)ds = \xi + \int\limits_{t_0}^t A(\xi + (s-t_0)A\xi)ds$$

$$= \xi + (t-t_0)A\xi + \int\limits_{t_0}^t (s-t_0)A^2\xi ds = \xi + (t-t_0)A\xi + A^2\xi\int\limits_{t_0}^t (s-t_0)ds$$

$$= \xi + (t-t_0)A\xi + \frac{(t-t_0)^2}{2}A^2\xi = \left(I + (t-t_0)A + \frac{(t-t_0)^2}{2}A^2\right)\xi$$

In general, we have

$$x_N(t) = \sum_{n=0}^N \frac{(t-t_0)^n}{n!}A^n\xi$$

and also, $\lim\limits_{N\to\infty} x_N(t) = \exp((t-t_0)A)\xi$ satisfies the Initial Value Problem.

Does this work for non-constant A?

Assume A is continuous. Then, define $x_{-1}(t)=0$ and $x_n(t)=\xi+\int\limits_{t_0}^{t}A(s)x_{n-1}(s)ds$, which is defined recursively. Also, define

$$\Delta x_n(t)=x_n(t)-x_{n-1}(t)$$

so that

$$\Delta x_n(t)=\int\limits_{t_0}^{t}A(s)\Delta x_{n-1}(s)ds \text{ for } n>0$$

so then $\Delta x_0(t)=\xi$ and

$$\Delta x_1(t)=\int\limits_{t_0}^{t}A(s)\Delta x_0(s)ds=\int\limits_{t_0}^{t}A(s)\xi ds$$

$$\Delta x(t)=\int\limits_{t_0}^{t}A(s)\Delta x_1(s)ds=\int\limits_{t_0}^{t}A(s)\int\limits_{t_0}^{r}A(r)\xi drds$$

so in general,

$$\Delta x_n(t)=\int\limits_{t_0}^{t}A(t_n)\int\limits_{t_0}^{t_n}A(t_{n-1})\dots\int\limits_{t_0}^{t_1}A(t_1)\xi dt_1\dots dt_n$$
$$=\int\limits_{t_0\le t_1\le\dots\le t_n\le t}\dots\int A(t_n)\dots A(t_1)\xi dt_1\dots dt_n$$

3.4.1. *Estimates (Inequalities).* Set

$$m=\max_{\alpha\le t\le\beta}||A(t)||\ \alpha\le t_0\le\beta$$

then we get

$$||\Delta x_n(t)||\le\frac{|t-t_0|^n}{n!}M^n||\xi|| \text{ for } \alpha\le t\le\beta$$

PROOF. By induction on $t\ge t_0$. For $n=0$ let $\Delta x_n(t)=\xi$, and for $n\ge 0$,

$$||\Delta x_n(t)||=\left|\left|\int\limits_{t_0}^{t}A(s)\Delta x_{n-1}(s)ds\right|\right|\le\int\limits_{t_0}^{t}||A(s)\Delta x_{n-1}(s)||ds$$

$$\le\int\limits_{t_0}^{t}||A(s)||\,||\Delta x_{n-1}(s)||ds\le\int\limits_{t_0}^{t}M\frac{(s-t_0)^{n-1}}{(n-1)!}M^{n-1}||\xi||ds$$

$$=\frac{M^n}{n!}(t-t_0)^n||\xi||$$

□

Also, $x_n(t)=\sum\limits_{j=0}^{n}\Delta x_j(t)$ so then $\sum\limits_{n=0}^{\infty}\frac{|t-t_0|^n}{n!}M^n||\xi||$ exists, and is equal to $\exp(|t-t_0|M)||\xi||$. Then by the comparison test, $\sum\limits_{n=0}^{\infty}\Delta x_n(t)$ exists and $||x(t)||\le\exp(|t-t_0|M)||\xi||$ and

$$x_n(t)=\xi+\int\limits_{t_0}^{t}A(s)x_{n-1}(s)ds$$

also,

$$x(t)=\xi+\int\limits_{t_0}^{t}A(s)x(s)ds$$

$$x'(t)=A(t)x(t)$$

We can use this to solve for the fundamental matrix $\frac{\partial}{\partial t}W(t_0,t)=A(t)W(t_0,t),\ W(t,t)=1$. Set $W_{-1}(t)=0$ then $W_n(t)=I+\int\limits_{t_0}^{t}A(s)W_{n-1}(t_0,s)ds$ which converges to a solution to the initial value problem.

4. Existence and Uniqueness

THEOREM 4.1 (Local Existence and Uniqueness for linear inhomogeneous ODE's). *If $A : [\alpha,\beta] \to \mathbb{R}^{m\times m}$ and $b : [\alpha,\beta] \to \mathbb{R}^m$ are continuous and $t_0 \in [\alpha,\beta],\ \xi \in \mathbb{R}^m$ then there is a unique solution $x : [\alpha,\beta] \to \mathbb{R}^m$ to the Initial Value Problem $x(t_0) = \xi,\ x'(t) = A(t)x(t) + b(t)$.*

PROOF. Construct W by Picard. $x(t) = W(t_0,t)\xi \times \int\limits_{t_0}^{t} W(s,t)b(s)ds$ works. □

THEOREM 4.2 (Global Existence and Uniqueness). *Same statement as local theorem, except with $\mathbb{R}$ in place of $[\alpha,\beta]$.*

PROOF. $\mathbb{R} = \cup_{L>0}[t_0 - L, t_0 + L]$ and apply the local theorem to $[t_0 - L, t_0 + L]$. □

What about non-linear examples?

$$x'(t) = F\left(t, x(t)\right),\ x(t_0) = \xi$$

If $F : S \to \mathbb{R}^m$ is continuous, for

$$S = \{(t,x) \in \mathbb{R}\times\mathbb{R}^m : |t - t_0| \le \delta,\ ||x - \xi|| \le r\} \text{ for } \delta, r > 0$$

(We will need more conditions later). Then

$$x_n(t) = \xi + \int\limits_{t_0}^{t} F\left(s, x_{n-1}(s)\right) ds,\ n > 0$$

$|t - t_0| \le \delta$ and $x(t_0) = \xi$ so then

$$x_1(t) = \xi + \int\limits_{t_0}^{t} F(s,\xi)ds,\ x_2(t) = \xi + \int\limits_{t_0}^{t} F(s,x(s))ds$$

Also, $\Delta x_n(t) = x_n(t) - x_{n-1}(t)$ for $n > 0$ and $x_n(t) = \xi + \sum\limits_{j=1}^{n} \Delta x_j(t)$,

$$\Delta x_n(t) = \int\limits_{t_0}^{t} \left(F(s, x_{n-1}(s)) - F(s, x_{n-2}(s))\right)$$

Problem: If F is non-linear in x, how is $F(s, x_{n-1}(s)) - F(s, x_{n-2}(s))$ related to $x_{n-1}(s) - x_{n-2}(s)$?
Consider $y, z : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^m$ such that $||y(t) - \xi|| \le r,\ ||z(t) - \xi|| \le r$. We then estimate $F(t, y(t)) - F(t, z(t))$. The trick is to introduce $U(t,\tau) = \tau y(t) + (1-\tau)z(t)$ so that $U(t,1) = y(t),\ U(t,0) = z(t)$ then

$$F\left(t, U(t,1)\right) - F\left(t, U(t,0)\right) = \int\limits_{0}^{1} \frac{\partial}{\partial t} F\left(t, U(t,\tau)\right) d\tau$$

$$\frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial x}\frac{\partial U}{\partial \tau} = \frac{\partial F}{\partial x}\left(t, U(t,\tau)\right)\left(y(t) - z(t)\right)$$

so the matrix equation is

$$\frac{\partial F_j}{\partial t}\left(t, U(t,\tau)\right) = \sum_{k=1}^{m} \frac{\partial F_j}{\partial x_k}\left(t, U(t,\tau)\right)\left(y(t)_k - z(t)_k\right)$$

This needs that $\frac{\partial F_j}{\partial x_k}(t,x)$ continuous for $(t,x) \in S$. Then

$$F(t, y(t)) - F(t, z(t)) = \int\limits_{0}^{1} \frac{\partial F}{\partial x}\left(t, U(t,\tau)\right)\left(y(t) - z(t)\right) d\tau$$

Set

$$M = \max_{s} \left|\frac{\partial F}{\partial x}\right|$$

then

$$|F(t, y(t)) - F(t, z(t))| \le \int\limits_{0}^{1} \left| |\frac{\partial F}{\partial x}(t,U)\right| |\, ||y(t) - z(t)||\, |d\tau \le M\, ||y(t) - z(t)||\, |$$

$$\Delta x_n(t) = \int\limits_{t_0}^{t} \left(F(s, x_{n-1}(s)) - F(s, x_{n-2}(s))\right) ds,$$

so

$$||\Delta x_n(t)|| \le \int_{t_0}^{t} M\,||x_{n-1}(s) - x_{n-2}(s)||ds = M \int_{t_0}^{t}$$

We then aim to show by induction that

$$||\Delta x_n(t)|| \le \frac{NM^{n-1}}{n!}|t-t_0|^n,\ n>0,\ N = \max_S ||F||$$

$x_0(t) = \xi$ and $x_1(t) = \xi + \int_{t_0}^{t} F(s,\xi)d\xi$ where the integral is equal to $\Delta x_1(t)$. $||\Delta x_1(t)|| \le |t-t_0|N$ so

$$||\Delta x_n(t)|| \le \int_{t_0}^{t} M\,||\Delta x_{n-1}(s)||ds \le \int_{t_0}^{t} M\frac{M^{-2}N}{(n-1)!}|s-t_0|^{n-1}ds = \frac{M^{n-1}N}{n!}|t-t_0|^n$$

$$||x_n(t) - \xi|| \le \sum_{j=1}^{n} ||\Delta x_j(t)|| \le \sum_{j=1}^{n} \frac{M^{j-1}N}{j!}|t-t_0|^j \le \sum_{j=1}^{\infty} \frac{M^{n-1}N}{n!}|t-t_0|^n$$

so we have

$$||x_n(t) - \xi|| \le \frac{N}{M}\left(\exp(|t-t_0|M) - 1\right) \le r \text{ if } |t-t_0| \le \frac{1}{M}\log\left(1 + \frac{Mr}{N}\right)$$

then define $\tilde{\delta} = \min\left(\delta, \frac{1}{M}\log\left(1+\frac{Mr}{N}\right)\right)$ so for $|t-t_0| \le \tilde{\delta}$

$$||\Delta x_n(t)|| \le \frac{M^{n-1}N}{n!}|t-t_0|^n \text{ and } ||x_n(t) - \xi|| \le \frac{N}{M}\left(\exp(|t-t_0|M) - 1\right)$$

both of which are proved by induction.
So $x_n(t)$ converges to some $x(t)$, i.e,

$$x_n(t) = \xi + \int_{t_0}^{t} F(s, x_{n-1}(s))ds,\ x(t) = \xi + \int_{t_0}^{t} F(s, x(s))ds$$

and $x(t_0) = \xi,\ x'(t) = F(t, x(t))$. Picards method gives us a solution to the initial value problem in $[t_0, \tilde{\delta}, t_0 + \tilde{\delta}]$ which gives existence. For uniqueness,

$$||F(t,y(t)) - F(t,z(t))|| \le M\,||y(t) - z(t)|| \text{ if } ||y(t) - \xi|| \le r,\ ||z(t) - \xi|| \le r$$

Now suppose $y'(t) = F(t, y(t))$ and $z'(t) = F(t, z(t))$ so $||y(t_0) - \xi|| < r,\ ||z(t_0) - \xi|| < r$ and $||y(t) - \xi|| < r,\ ||z(t) - \xi|| < r$ for t near t_0, so

$$||F(s,y(s)) - F(s,z(s))|| \le M\,||y(s) - z(s)||$$

for s between t and t_0.

$$y(t) = y(t_0) + \int_{t_0}^{t} F(s, y(s))ds,\ z(t) = z(t_0) + \int_{t_0}^{t} F(s, z(s))ds$$

$$y(t) - z(t) = y(t_0) - z(t_0) + \int_{t_0}^{t} \left[F(s,y(s)) - F(s,z(s))\right] ds$$

and

$$||(t) - z(t)|| = ||(t_0) - z(t_0)|| + M\int_{t_0}^{t} ||y(s) - z(s)||ds$$

This estimate is valid on some interval $I = [t_0 - \tilde{\delta}, t_0 + \tilde{\delta}]$ for $\tilde{\delta} > 0$. Let

$$C = \max_{t \in I} \frac{||y(t) - z(t)||}{||y(t_0) - z(t_0)||\,||\exp(|t-t_0|M)||}$$

then

$$||y(t) - z(t)|| = C\,||y(t_0) - z(t_0)||\,||\exp(|t-t_0|M)|| \text{ for some } t \in I$$

$$||y(s) - z(s)|| \le C\,||y(t_0) - z(t_0)||\exp(|s-t_0|M)\ \forall s \in I$$

$$||y(t) - z(t)|| \le ||y(t_0) - z(t_0)|| + CM\int_{t_0}^{t} ||y(t_0) - z(t_0)||\exp(|s-t_0|M)ds$$

$$= ||y(t_0) - z(t_0)|| \left(1 + CM \int_{t_0}^{t} \exp(|s - t_0|M) ds \right)$$

$$= ||y(t_0) - z(t_0)|| \, [1 + C[\exp(|t - t_0|M) - 1]] = C \, ||y(t_0) - z(t_0)|| \exp(|t - t_0|M) - (C - 1) \, ||y(t_0) - z(t_0)||$$

so $(C-1) \, ||y(t_0) - z(t_0)|| \leq 0$ so $C \leq 1$. This means, $\forall t$,

$$||y(t) - z(t)|| \leq ||y(t_0) - z(t_0)|| \exp(|t - t_0|M)$$

COROLLARY 4.3. Uniqueness of Solutions: *Take* $y(t_0) = z(t_0) = \xi$. *Then* $||y(t) - z(t)|| \leq 0 \Rightarrow \; y(t) = z(t)$.

COROLLARY 4.4. Continuous dependence on initial conditions $x(t)$ *depends continuously on* ξ.

So we can formulate these results more concisely.

THEOREM 4.5 (Picard Existence and Uniqueness theorem for ODE's). *If* $F, \frac{\partial F}{\partial x}$ *continuous on* S *then there is a unique solution to*

$$x'(t) = F(t, x(t)), \; x(t_0) = \xi$$

for $|t - t_0| \leq \tilde{\delta}$ *depending continuously on* ξ.

4.1. More Existence and Uniqueness Theorems.

THEOREM 4.6 (Theorem 0). *If* F *continuous and* $\frac{\partial F_j}{\partial x_k}$ *is continuous near* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^m$ *then the IVP* $x'(t) = F(t, x(t)), \; x(t_0) = x_0$ *has a unique solution near* t_0 *and depends continuously on* x_0 *(and* t_0*).*

4.1.1. *Improvements.*

THEOREM 4.7 (Theorem 1). *If* $G : U \subseteq \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ *is continuous in some neighbourhood* U *of* (t_0, x_0, v_0) *and* $\frac{\partial G_j}{\partial x_k}, \frac{\partial G_j}{\partial v_k}$ *also continuous and* $\frac{\partial G_j}{\partial v_k}$ *is invertible the the IVP* $G(t, x(t), x'(t)) = 0, \; x(t_0) = x_0, \; x'(t_0) = v_0$ *has a unique solution near* t_0 *if* $G(t_0, x_0, v_0) = 0$.

PROOF. Theorem 0 and the Implicit Function Theorem. ☐

Example $x'(t)^3 - x(t)^2 = 0$ in $\mathbb{R}$. Then $G(t, x, v) = v^3 - x^2$ and $G, \frac{\partial G}{\partial x} = -2x$ and $\frac{\partial G}{\partial v} = 3v^2$ are all continuous and $\frac{\partial G}{\partial v}$ invertible if and only if $v \neq 0$. $x'(t_0) = v_0$ has a unique solution near t_0 provided

$$v_0^3 - x_0^2 = 0 \; v_0 \neq 0$$

$$v_0 = x_0^{\frac{2}{3}} \; x_0 \neq 0$$

So $x'(t)^3 = x(t)^2$ gives $x'(t) = x(t)^{\frac{2}{3}}, \; x(t_0) = x_0, \; F(t, x) = x^{\frac{2}{3}}$ so it has a unique solution near t_0 if $x_0 \neq 0$. This is the example we did in week 2 with $\frac{dy}{dx} = y^{\frac{2}{3}}$ which was separable.

THEOREM 4.8 (Theorem 2). *If* F *is continuous near* (t_0, x_0, ω) *in* $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ *along with* $\frac{\partial F_j}{\partial x_k}$ *then there is a unique solution to* $x'(t) = F(t, x(t)\omega), \; x(t_0) = x_0$ *which depends continuously on* t_0, x_0, ω.

Example $x'' + kx = 0, \; x' = v$ so $v' = -kx$ with $x(0) = 0, \; v(0) = 1$. An explicit solution is

$$x(t) = \begin{cases} \frac{\sin(\sqrt{k}t)}{\sqrt{k}} & k > 0 \\ k & k = 0 \\ \frac{\sin(\sqrt{-k}t)}{\sqrt{-k}} & k < 0 \end{cases}$$

and continuity at $k = 0$ can be shown by L'Hopital.

PROOF OF THEOREM 2. Promote parameters to dependent variables, giving:

$$x_{m+1}(t_0) = \omega_1, \ldots, x_{m+n}(t_0) = \omega_n$$

$$x'_{m+1}(t) = 0, \ldots, x'_{m+n}(t) = 0$$

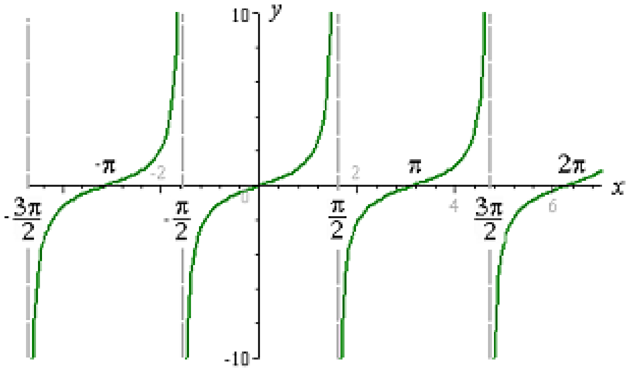
and then apply Theorem 0 to the extended IVP. ☐

THEOREM 4.9 (Theorem 3). *If* $(t_0, x_0) \in U \subseteq \mathbb{R} \times \mathbb{R}^m$ *and* $F : U \to \mathbb{R}^m$ *is continuous along with* $\frac{\partial F_j}{\partial x_k}$ *in* U*, then there is a unique* <u>*maximally extended*</u> *solution to* $x'(t) = F(t, x(t)), \; \frac{dx}{dt} = 0, \; x(t_0) = x_0$ *for some* $a, b, \; a < t_0 < b$ *and* $x(t) = x_0$ *for* $a < t < b$ *is a solution. For* $A < a < t_0 < b < B$*, define* $\tilde{x}(t) = x0$ *for* $A < t < B$. *Then* $x(t) = x_0$ *for* $t \in \mathbb{R}$ *is a maximally extended solution.*

THEOREM 4.10 (Theorem 4). *There is a maximally extended solution to* $x'(t) = F(t, x(t)), \; x(t_0) = x_0$.

PROOF. Pretty much just an exercise in logic. Extend until you can't extend any more. Roughly, we can extend until:

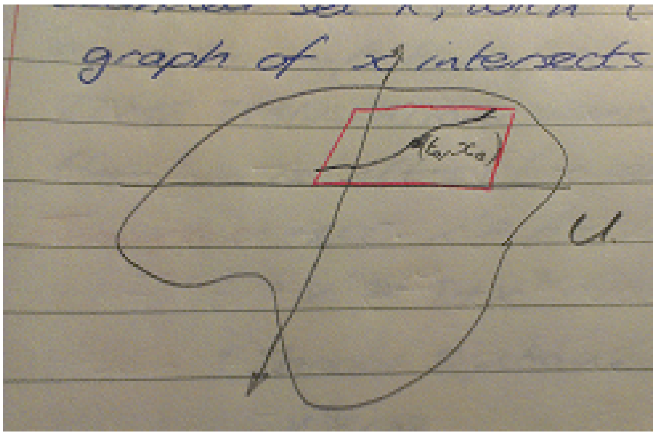
- We leave U.
- The solution goes to ∞. For example, $x'(t) = 1 + x(t)^2$ has solution $x(t) = \tan(t - s)$ for some s, which goes to infinity as can be seen by graphing $\tan$.



- The domain of definition is $\mathbb{R}$, i.e a global solution.

□

THEOREM 4.11 (Theorem 4). *If $(t_0, x_0) \in U$, $U \subseteq \mathbb{R} \times \mathbb{R}^m$ open and $F : U \to \mathbb{R}^m$, $\frac{\partial F}{\partial x}$ are continuous in U then there is a solution to the IVP such that for any closed bounded set K with $(t_0, x_0) \in K \subseteq U$ the graph of x intersects the boundary of K.*



Example Jacobi

$$x'(t) = y(t)z(t)$$

$$y'(t) = -x(t)z(t)$$

$$z'(t) = -k^2x(t)y(t)$$

for $k \neq 0$. Note this can solved explicitly for $k = 0$ as it is a linear constant coefficient equation.
$x^2 + y^2$ and $k^2x^2 + z^2$ are invariants. Also

$$x(t)^2 + y(t)^2 + z(t)^2 \leq x(t)^2 + y(t)^2 + k^2x(t)^2 + z(t)^2 = x(s)^2 + y(s)^2 + k^2x(s)^2 + z(s)^2$$

for some s. Then this is

$$\leq x(s)^2 + y(s)^2 + z(s)^2 + k^2x(s)^2 + k^2y(s)^2 + k^2z(s)^2$$

$$\Rightarrow x(t)^2 + y(t)^2 + z(t)^2 \leq (1 + k^2)(x(s)^2 + y(s)^2 + z(s)^2)$$

or equivalently,

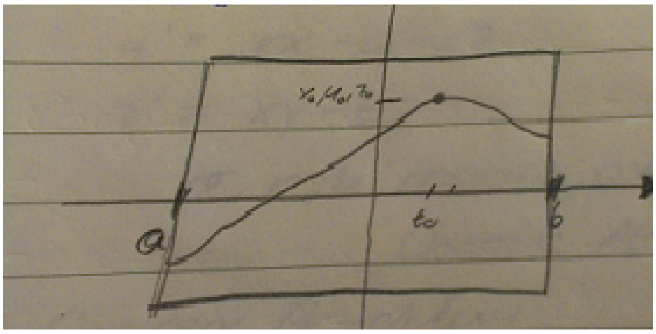
$$||x(t), y(t), z(t)|| \leq (1 + k^2)^{\frac{1}{2}}||x(s), y(s), z(s)||$$

For x_0, y_0, z_0, t_0 given, choose $R > (1 + k^2)^{\frac{1}{2}}||x_0, y_0, z_0||$. **Define**

$$\kappa = \{(t, x, y, z) \in \mathbb{R}^4 : a \leq t \leq b \text{ and } ||x, y, z|| < R\}$$

for $a \leq t_0 \leq b$, which has the following diagram: which must leave through the sides of the rectangle. We can do this for any a, b so there is a global solution to the IVP.

We can combine the ideas from Theorem 1,2,3 into a huge theorem, but there is no point.



5. Autonomous Systems

DEFINITION 5.1. *An autonomous system is a differential equation of the form*

$$x' = F(x)$$

(no t dependence).

DEFINITION 5.2. *Time translation invariance means if $x(t)$ is a solution then so is $x(t-s)$ for any s.*

Example Linear constant coefficient.

$$x' = Ax$$

Example Jacobi System.

$$x' = yz \; y' = -xz \; z' = -k^2xy$$

Example Van Der Pol.

$$x'' + \mu(x^2-1)x' + x = 0$$

so $x' = v$ and $v' = \mu(1-x^2)v - x$.

Example Predator-Prey.

$$N' = rN(1-\frac{N}{k}) - aNP$$
$$P' = -cP + bNP$$

where a, b, c, k, r are positive parameters, P is the number of predators and N is the number of prey.

Example Lorenz System.

$$x' = \sigma(y-x) \; y' = rx - y - xz \; z' = xy - bz$$

for σ, r, b positive parameters. First studied example of a chaotic system.

Example Damped Pendulum.

$$\theta'' + k\sin(\theta) + \lambda(\theta')\theta' = 0$$

where $k \geq 0$ a constant and $\lambda \geq 0$ a function.

5.1. Equilibrium.

DEFINITION 5.3. *ψ is an equilibrium of*

(9) $$x'(t) = F(x(t))$$

if $F(\psi) = 0$ or equivalently, if $x(t) = \psi$ is a solution.

Example Pendulum, undamped.

$$\theta'' + k\sin\theta = 0 \; k > 0$$

So $\theta' = \omega, \; \omega' = -k\sin\theta$ or equivalently,

$$F\left(\begin{pmatrix} \theta \\ \omega \end{pmatrix}\right) = \begin{pmatrix} \omega \\ -k\sin\theta \end{pmatrix}$$

Equilibria occur where $\omega = 0$ i.e where $k\sin\theta = 0$, so where $\theta = n\pi$ for some integer n. There are only really 2 cases, where n is even (hanging straight down) and odd(straight up).

DEFINITION 5.4. *ψ is an unstable equilibrium if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $x'(t) = F(x(t)) \; x(t_0) = x_0$ then*

$$||x_0 - \psi|| < \delta \; \Rightarrow \; ||x(t) - \psi|| < \epsilon$$

for all $t > t_0$. Otherwise, ψ is an unstable equilibrium.

For the pendulum, the quantity $\omega^2 + k\cos\theta$ is invariant.

DEFINITION 5.5. *ψ is a strictly stable equilibrium of (9) if it's stable and $\lim_{t\to+\infty} x(t) = \psi$ provided $||x_0 - \psi|| < \delta$.*

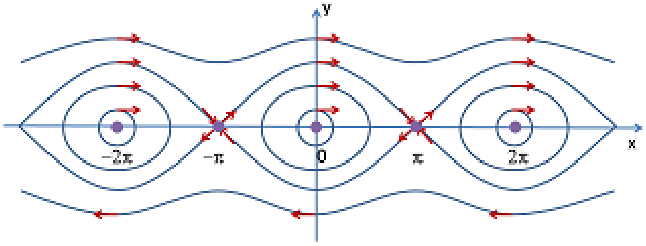


FIGURE 2. Phase Portrait for Pendulum

For undamped pendulum, $\theta'' + k\sin\theta = 0$, then $(\theta, \theta') = (0, 0)$ is stable, but not strictly stable.
For damped pendulum,$\theta''+k\sin\theta+\lambda(\theta')\theta' = 0,\ \lambda > 0$, $(0, 0)$ is a stable equilibrium and $(\pi, 0)$ is always unstable.

Example Linear Constant Coefficient $x' = Ax$. ψ is an equilibrium if and only if $A\psi = 0$, so if $\psi \in N(A)$. If A is invertible then 0 is the only equilibrium.

Example Jacobi

$$x' = yz\ y' = -xz\ z' = -k^2xy$$

$(0, 0, 0)$ is an equilibrium, and in fact it is sufficient for any two of the functions to be 0, so any point on the coordinate axes.

Example Van Der Pol

$$x' = v\ v' = \mu(1 - x^2)v - x$$

$(0, 0, 0)$ is an equilibrium, and is the only one.

Example Predator-Prey

$$N' = rN(1 - \frac{N}{k}) - aNP\ P' = -cP + bNP = P(-c + bN)$$

Then $(N, P) = (0, 0)$ is an equilibrium, along with $(k, 0)$. Similarly, letting $N = \frac{-c}{b}$ gives $0 = rN(1 + \frac{c}{bk} - aP)$ giving $P = \frac{1+\frac{c}{bk}}{a}$ so $(\frac{-c}{b}, \frac{1+\frac{c}{bk}}{a})$ is another solution.

Example Lorenz

$$x' = \sigma(y - x)\ y' = rx - y - xz\ z' = xy - bz$$

So $(0, 0, 0)$ is an equilibrium. Also, letting $x = y$ gives $z = \frac{x^2}{b}$ and

$$rx - x - \frac{x^3}{b} = x^3 - b(1 - r)x = 0 \Rightarrow\ x = \pm\sqrt{b(1 - r)} = y\ \Rightarrow\ z = 1 - r$$

5.2. Stability.

Example Jacobi

$$||x(t), y(t), z(t)|| \le \sqrt{1 + k^2}||x_0, y_0, z_0||$$

Then choose $\delta = \frac{\epsilon}{\sqrt{1+k^2}}$ and

$$||(x_0, y_0, z_0) - (0, 0, 0)|| < \delta\ \Rightarrow\ ||(x(t), y(t), z(t)) - (0, 0, 0)|| < \epsilon$$

for all $t > t_0$, i.e $(0, 0, 0)$ is a stable equilibrium but it is not strictly stable. For example, $||(0, \frac{\delta}{2}, 0) - (0, 0, 0)|| < \delta$ but the solution to the IVP $(x_0, y_0, z_0) = (0, \frac{\delta}{2}, 0)$ does not go to $(0, 0, 0)$.

5.3. Invariants and Stability for Autonomous Systems.

For the system

$$x'(t) = F(x(t))$$

I is an invariant if $\frac{dI}{dt} = 0$ for all solutions. By the chain rule,

$$\frac{dI(x(t))}{dt} = \sum_{j=1}^{m} \frac{\partial I}{\partial x_j}(x)F_j(x)$$

and I is invariant if and only if

$$\sum_{j=1}^{m} \frac{\partial I}{\partial x_j}(x)F_j(x) = 0$$

Note: The existence of an invariant does not imply stability.

Example

$$x' = y \; y' = x$$

The invariant is $x^2 + y^2 = I$ and $\frac{dI}{dt} = 2xx' - 2yy' = 2xy - 2yx = 0$. $(0,0)$ is an equilibrium but it is not stable.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

so

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp(t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

This is stable if for all $\epsilon > 0$ there is a $\delta > 0$ such that $||(x_0.y_0)|| < \delta \;\Rightarrow ||x(t), y(t)|| < \epsilon$, so all solutions with initial data near $(0,0)$ are bounded.
When is $(x(t), y(t))$ bounded?

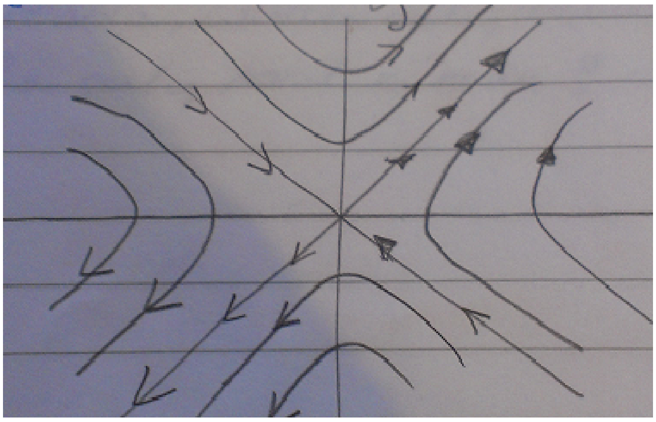
$$\|x(t), y(t)\|^2 = (x(t), y(t)) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (x(t), y(t)) \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= (x_0, y_0) \begin{pmatrix} \cosh(2t) & \sinh(2t) \\ \sinh(2t) & \cosh(2t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \frac{1}{2}(x_0, y_0) \begin{pmatrix} e^{2t} & e^{2t} \\ e^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \frac{1}{2}(x_0, y_0) \begin{pmatrix} e^{-2t} & e^{-2t} \\ e^{-2t} & e^{-2t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \frac{1}{2} e^{2t}(x_0 + y_0)^2 + \frac{1}{2} e^{-2t}(x_0 - y_0)^2$$

which is unbounded if $x_0 + y_0 \neq 0$, so $(0,0)$ is unstable.



THEOREM 5.6. *If $U \subseteq \mathbb{R}^m$ is open, $\psi \in U$, $F : U \to \mathbb{R}^m$ continuously differentiable and $I : U \to \mathbb{R}$ also continuously differentiable. If I has a strict local minimum at ξ then ξ is a stable equilibrium but ξ is not strictly stable.*
Note: *We didn't assume that ξ is an equilibrium.*

Example
- $x' = y, \; y' = -x$ which has invariant $x^2 + y^2$ and has a strict local minimum at $(0,0)$ so this is a stable equilibrium.
- Jacobi System.

$$x' = yz \; y' = -xz \; z' = -k^2 xy$$

  which has invariants $x^2 + y^2, \; k^2x^2 + z^2, \; (1 + k^2)x^2 + z^2$.

PROOF. ξ is strict local minimum of I and there is an $R > 0$ such that

$$0 < ||x - \xi|| < R \;\Rightarrow I(x) > I(\xi)$$

then choose $r > 0$ such that $r < \min(R, \epsilon)$. Define

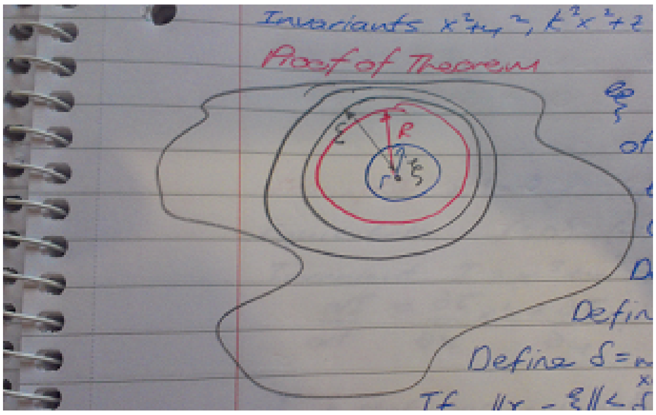
$$J = \min_{||x-\xi||=r} I(x)$$

then $J > I(\xi)$. Define

$$K = \{x \in U : ||x - \xi|| < R \text{ and } I(x) = J\}$$

and also define $\delta = \min_{x \in K} ||x - \xi||, \; \delta > 0$. If $||x_0 - \xi|| < \delta$ then $I(x(t)) = I(x_0)$ for all $t > t_0$ so $I(x(t)) < J$ and $||x(t) - \psi|| < \epsilon$ so ξ isn't strictly stable.
Strictly stable is stable and $\lim_{t\to\infty} x(t) = \xi$ whenever $||x_0 - \xi|| < \delta$. I is continuous so

$$\lim_{t\to\infty} I(x(t)) = I\left(\lim_{t\to\infty} x(t)\right) = I(\xi)$$



if ξ is strictly stable, which only happens if $I(x_0) = I(\xi)$. The only point x_0 near ξ with $I(x_0) = I(\xi)$ is $x_0 = \xi$. $\square$

Example

$$x' = 2y \; y' = 3x^2 - 3$$

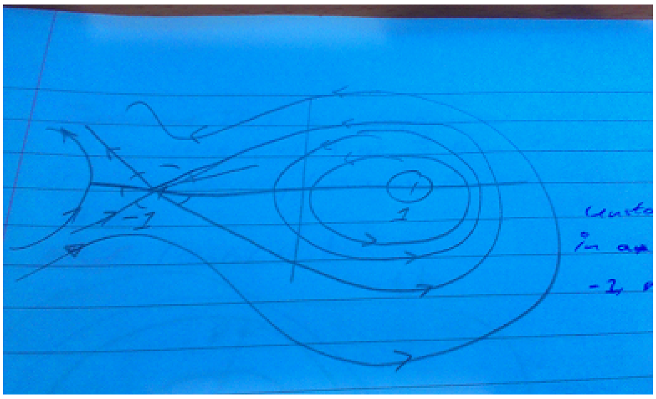
Which has equilibria at $(1, 0), (-1, 0)$ and has invariant $I = x^3 + y^2 - 3x$ so

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}x' + \frac{\partial I}{\partial y}y' = -(3x^2 - 3)2y + (3x^2 - 3)2y = 0$$

I is stationary where $\frac{\partial I}{\partial x} = 3x^2 - 3 = 0$ and where $\frac{\partial I}{\partial y} = 2y = 0$ so the stationary points are $(1, 0)$ and $(-1, 0)$ also. Looking at the Hessian:

$$\begin{pmatrix} \frac{\partial^2 I}{\partial x^2} & \frac{\partial^2 I}{\partial x \partial y} \\ \frac{\partial^2 I}{\partial y \partial x} & \frac{\partial^2 I}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}$$

which is positive definite at $(1, 0)$ and indefinite at $(-1, 0)$ so $(1, 0)$ is a strict minimum of I, while $(-1, 0)$ is a saddle point. So $(1, 0)$ is a stable equilibrium but not strictly stable, while $(-1, 0)$ is an unstable equilibrium.



DEFINITION 5.7. *With U, F as before, $\xi \in U$ we say*

$$L : U \to \mathbb{R}$$

is a Lyapunov function of

$$x'(t) = F(x(t)) \text{ at } \xi$$

if:

- *L is continuously differentiable*
- *L has a strict local minimum at ξ*
- *There is some $\rho > 0$ such that if*

$$||x - \xi|| < \rho \text{ then } \sum_{j=1}^{m} \frac{\partial L}{\partial x_j}(x) F_j(x) \leq 0$$

L is called a strict Lyapunov function if

$$0 < ||x - \xi|| < \rho \Rightarrow \sum_{j=1}^{m} \frac{\partial L}{\partial x_j}(x) F_j(x) < 0$$

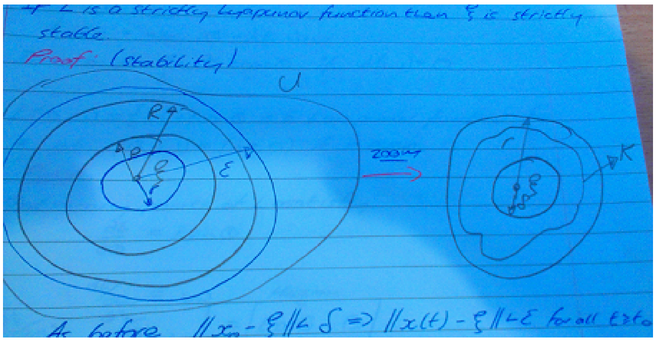
Note: If x is a solution to $x' = F(x)$ then

$$\frac{d}{dt}F(x(t)) = \sum_{j=1}^{m} \frac{\partial L}{\partial x_j}(x(t))F_j(x(t))$$

and $\sum_{j=1}^{m} \frac{\partial L}{\partial x_j}F_j < 0$ implying $L(x)$ is strictly increasing.

Note: If I is an invariant and has a strict local minimum at ξ then I is a Lyapunov function, but not a strict Lyapunov function.

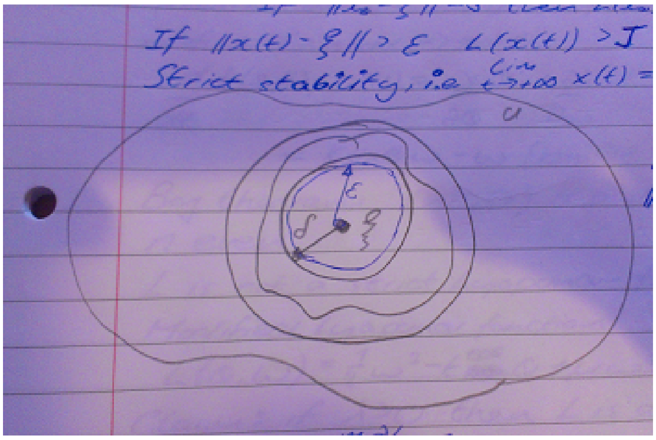
THEOREM 5.8. *If L is a Lyapunov function for $x'(t) = F(x(t))$ at ξ then ξ is a stable equilibrium. If L is a strictly Lyapunov function then ξ is strictly stable.*



PROOF. Prove stability first.
As before,

$$||x_0 - \xi|| < \delta \ \Rightarrow \ ||x(t) - \xi|| < \epsilon$$

for all $t > t_0$. $\delta < \min_{x \in K} ||x - \xi||, \ \delta > 0$. If $||x_0 - \xi|| < \delta$ then $L(x_0) < J$. If $||x(t) - \xi|| < \epsilon, \ L(x(t)) > J$.
Then prove strict stability.
If $||x_0 - \xi|| < \delta$ then $||x(t) - \xi|| < r$ for all $t > t_0$. So $\sum_{j=1}^{m} \frac{\partial L}{\partial x_j}F_j < 0$. Set



$$z = \max_{\epsilon < ||x - \xi|| < r} \frac{\partial L}{\partial x_j}F_j(x)$$

So $\frac{d(L(x(t)))}{dt} < z < 0$ and

$$L(x(t)) < L(x_0) - (t - t_0)z \ \rightarrow -\infty \text{ as } t \rightarrow +\infty$$

as required. □

Example Damped Pendulum

$$\theta'' + k\sin\theta + \lambda(\theta')\theta' = 0 \ k > 0$$

(10) $$\theta' = \omega \ \omega' = -k\sin\theta - \lambda(\omega)\omega$$

Which has invariant $I = \frac{1}{2}\omega^2 - k\cos\theta$ if $\lambda(\omega) = 0$. $L(\theta, \omega) = \frac{1}{2}\omega^2 - k\cos\theta$ is a Lyapunov function for (10) at $(\theta, \omega) = (n\pi, 0)$ if n is even.

PROOF. The stationary points of L satisfy

$$\frac{\partial L}{\partial \theta} = k \sin\theta$$
$$\frac{\partial L}{\partial \omega} = \omega$$

then looking at the Hessian,

$$\begin{pmatrix} \frac{\partial^2 L}{\partial \theta^2} & \frac{\partial^2 L}{\partial \theta \partial \omega} \\ \frac{\partial^2 L}{\partial \omega \partial \theta} & \frac{\partial^2 L}{\partial \omega^2} \end{pmatrix} = \begin{pmatrix} k\cos\theta & 0 \\ 0 & 1 \end{pmatrix}$$

which is positive definite at $(n\pi, \omega)$ for n even which is a strict local minimum and is indefinite for n odd, which is a saddle point.

$$\frac{d}{dt} L(\theta(t), \omega(t)) = \frac{\partial L}{\partial \theta}\theta' + \frac{\partial L}{\partial \omega}\omega'$$
$$= k\sin\theta\omega - \omega(k\sin\theta + \lambda(\omega)\omega) = -\lambda(\omega)\omega^2 \leq 0$$

So then, by the theorem, $(n\pi, 0)$ is a stable equilibrium for n even.
L is not a strict Lyapunov function. Take the modified Lyapunov function

$$L(\theta, \omega) = \frac{1}{2}\omega^2 - k\cos\theta + \mu\omega\sin\theta$$

Claim: If $\lambda > 0$ then L is a strict Lyapunov function for (10) at $(n\pi, 0)$ and n odd.

$$\frac{\partial L}{\partial \theta} = k\sin\theta + \mu\omega\cos\theta \; \frac{\partial L}{\partial \omega} = \omega + \mu\sin\theta$$

which is still zero at $(n\pi, 0)$. Then getting the components of the Hessian,

$$\frac{\partial^2 L}{\partial \theta^2} = k\cos\theta - \mu\omega\sin\theta \; \frac{\partial^2 L}{\partial \theta \partial \omega} = \mu\cos\theta \frac{\partial^2 L}{\partial \omega^2} = 1$$

$$\begin{pmatrix} k\cos\theta - \mu\omega\sin\theta & \mu\cos\theta \\ \mu\cos\theta & 1 \end{pmatrix}$$

which is positive definite for sufficiently small μ.

$$\frac{\partial L}{\partial t}(\theta(t), \omega(t) = \frac{\partial L}{\partial \theta}\theta' + \frac{\partial L}{\partial \omega}\omega'$$
$$= (k\sin\theta + \mu\omega\cos\theta)\omega - (\omega + \mu\sin\theta)(k\sin\theta + \lambda(\omega)\omega)$$
$$= k\omega\sin\theta + \mu\omega^2\cos\theta - k\omega\sin\theta - \lambda(\omega)\omega^2 - k\mu\sin^2\theta - \lambda(\omega)\mu\omega\sin\theta$$

We can rewrite this using the following substitutions:

$$a(\theta, \omega) = k\mu \; b(\theta, \omega) = \lambda(\omega)\mu \; c(\theta, \omega) = \lambda(\omega) - \mu\sin\theta$$

giving $ax^2 + bxy + cy^2$ which is positive definite for small μ.

$$\Delta(\theta, \omega) = b^2 - 4ac = \lambda^2\mu^2 - 4k\lambda\mu + 4k\mu^2\cos\theta < 0$$

for small positive μ. $\frac{dL}{dt} < 0$ unless $(\omega, \sin\theta) = (0, 0)$. For small positive μ, L is a strict Lyapunov function for (10) at $(n\pi, 0)$ for n even. So these are strictly stable. □

5.4. Stability of Linear Autonomous Systems. Suppose $x' = Ax, \; x = 0$ is an equilibrium. Is it stable? Is it strictly stable?
Method 1:
Quadratic Lyapunov functions

$$(x^T)' = x^T A^T$$

If $L(x) = x^T Bx, \; B = B^T$ then

$$\frac{d}{dt} L(x(t)) = \left(x^T\right)' Bx + x^T Bx' = x^T A^T Bx + x^T BAx$$
$$= x^T \left(A^T B + BA\right) x = -x^T Cx$$

say. So then,

$$A^T B + BA + C = 0$$

and also

$$\frac{d}{dt} L(x(t)) \leq 0$$

if C is positive semi-definite and

$$\frac{d}{dt} L(x(t)) < 0$$

if C is positive definite (assuming $x(t) \neq 0$).

THEOREM 5.9. *0 is a strict stable equilibrium of $x' = Ax$ if there are positive definite matrices B, C such that $A^T B + BA + C = 0$.*
0 is a stable equilibrium if there is a positive definite B and a positive semi-definite C such that

$$A^T B + BA + C = 0$$

PROOF. $L(x) = x^T Bx$ is a strict Lyapunov function in first case, Lyapunov in second case. □

Method 2: Just solve $x' = Ax$.

$$x(t) = \exp(tA)\, x_0$$

and then make a linear change of variable with $y = V^{-1}x$

$$y' = V^{-1}x' = V^{-1}Ax = V^{-1}AVy$$

By the assignment (5) 0 is a (strictly) stable equilibrium of $x' = Ax$ if and only if 0 is a (strictly) stable equilibrium of $y' = V^{-1}AVy$.
We can always choose V so that $J = V^{-1}AV$ is in Jordan normal form.

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & J_l \end{pmatrix} \text{ where } J_k = \begin{pmatrix} \lambda_k & 1 & \dots & 0 \\ 0 & \lambda_k & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \dots & 0 & \lambda_k \end{pmatrix}$$

each block of size $m_k \times m_k$.

$$y' = Jy \;\Rightarrow\; y(t) = \exp(tJ)y_0$$

$$\exp(tJ) = \begin{pmatrix} \exp(tJ_1) & 0 & 0 \\ & \ddots & \vdots \\ 0 & \dots & \exp(tJ_l) \end{pmatrix}$$

and then

$$\exp(tJ_k) = \begin{pmatrix} \exp(t\lambda_1) & t\exp(t\lambda_k) & \dots & t^{m_k-1}\frac{\exp(\lambda_k t)}{(m_k-1)!} \\ & \ddots & \ddots & \\ 0 & & \dots & \exp(t\lambda_k) \end{pmatrix}$$

then we have $||y(t)|| \leq ||\exp(tJ)||||y_0||$ so $||\exp(tJ)||$ bounded for $t \geq 0$ which implies stability. Similarly, $||\exp(tJ)|| \to 0$ implies strict stability. Then,

$$\|\exp(tJ)\|^2 = tr\left(\overline{\exp(tJ)^t}\exp(tJ)\right)$$

and

$$\overline{\exp(tJ_1)^T}\exp(tJ) = \begin{pmatrix} \overline{\exp(tJ)^T}\exp(tJ_1) & 0 & \dots \\ 0 & \ddots & \vdots \\ 0 & \dots & \overline{\exp(tJ_l)^T}\exp(tJ_l) \end{pmatrix}$$

and

$$\|\exp(tJ)\| = \sum_{k=1}^{l} tr\left(\overline{\exp(tJ_k)^T}\exp(tJ_k)\right) = \sum_{k=1}^{l}\|\exp(tJ_k)\|^2$$
$$= \sum_{k=1}^{l}|\exp(\lambda_k t)|^2\sum_{j=0}^{m_k-1}\left(\frac{t^j}{j!}\right)^2(m_k - j)$$

and then

$$|\exp(\lambda_k t)|^2 = \overline{\exp(\lambda_k t)}\exp(\lambda_k t) = \exp(\overline{\lambda_k} t)\exp(\lambda_k t)$$
$$= \exp((\lambda_k + \overline{\lambda_k} t)) = \exp(2Re\lambda_k t)$$

So then for:

$$|\exp(\lambda_k t)|^2\sum_{j=0}^{m_k-1}\frac{t^{2j}}{j!}(m_k - j)$$

If $\lambda < 0$ then this is bounded and goes to zero. If $Re\lambda_k = 0$ and $m_k = 1$ then bounded, otherwise not bounded.

DEFINITION 5.10. *The geometric multiplicity of an eigenvalue λ of A is $dim(N(A - \lambda I))$. The algebraic multiplicity is the order of vanishing of the characteristic polynomial of A at λ.*

We then have

$$1 \leq \text{ Geometric Multiplicity } \leq \text{ Algebraic Multiplicty}$$

and the geometric multiplicity is equal to the algebraic multiplicity if and only if there are no off diagonal $1's$ in the Jordan blocks for λ.

THEOREM 5.11. *If all eigenvalues λ of A have $Re\lambda < 0$ then 0 is a strictly stable equilibrium of $x' = Ax$. If all eigenvalues λ satisfy $Re\lambda \leq 0$ and either $Re\lambda < 0$ or algebraic multiplicity of λ is equal to geometric multiplicity then 0 is a stable equilibrium.*

PROOF. See previous calculation of $\|\exp(tJ)\|$. □

Example General 2×2 case.

$$u' = \alpha u + \beta v \; v' = \gamma u + \delta v$$

Then $x' = Ax$ with

$$x = \begin{pmatrix} u \\ v \end{pmatrix} \; A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

When is 0 stable? Strictly stable?
Characteristic polynomial: $\lambda^2 - trA\lambda + \det A$, which has roots (eigenvalues) $\lambda = r \pm \sqrt{\Delta}$, $r = \frac{1}{2}trA$, $\Delta = (trA)^2 - 4\det A = 4(r^2 - \det A)$. We then consider 4 cases:
Case 1: $\Delta > 0$ so distinct real roots $\lambda_+, \lambda_- = r \pm \sqrt{\Delta}$. Then $\lambda_+ + \lambda_- = trA = 2r,\; \lambda_+\lambda_- = \det A$.

$$\lambda_+, \lambda_- < \Leftrightarrow r < 0, \det A > 0$$
$$\lambda_+, \lambda_- \leq 0 \Leftrightarrow r < 0, \det A \geq 0$$

Case 2:$\Delta < 0$ so has complex conjugate eigenvalues $r \pm i\sqrt{\Delta}$. For both, $Re\lambda = r$.

$$Re\lambda < 0 \Leftrightarrow r < 0,\; Re\lambda \leq 0 \Leftrightarrow r \leq 0$$

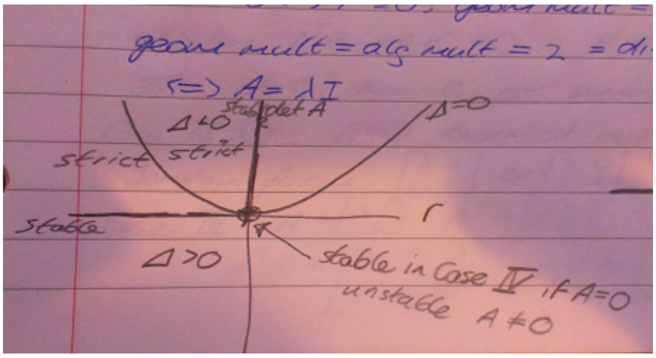
Case 3: $\Delta = 0$ so double real root $\lambda = r$. If the geometric multiplicity is 1 then

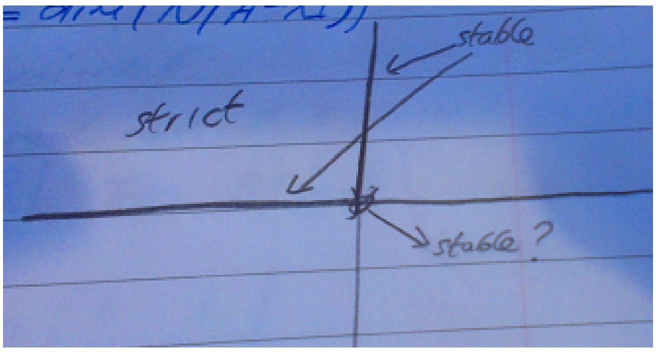
$$Re\lambda < 0 \Leftrightarrow r < 0 \text{ strict stability}$$

and $Re\lambda \leq 0 \Leftrightarrow r \leq 0$ but algebraic multiplicity is not equal to geometric multiplicity so $r = 0$ is not stable.
Case 4:$\Delta = 0$ so $\lambda = r$ is a double root.

$$Re\lambda < 0 \Leftrightarrow r < 0 \text{ strictly stable}$$

and $Re\lambda \leq 0 \Leftrightarrow r \leq 0$ and geometric multiplicity equal to algebraic multiplicity so stable. Geometric multiplicity is equal to algebraic which is equal to $2 = dim(N(A - \lambda I)) \Leftrightarrow A = \lambda I$.





THEOREM 5.12. *There are positive definite B, C such that $A^T B + BA + C = 0$ if and only if all eigenvalues λ satisfy $Re\lambda < 0$.*
There are positive definite B positive semi-definite C such that $A^T B + BA + C = 0$ if and only if for all eigenvalues $\lambda,\ Re\lambda \leq 0$ and either $Re\lambda < 0$ or the geometric multiplicity is equal to the algebraic multiplicity.

PROOF. We will only outline it and then prove the 2×2 case.First, we can replace A by any similar matrix $\tilde{A} = V^{-1}AV$. Suppose $\tilde{A}^T\tilde{B} + \tilde{B}\tilde{A} + \tilde{C} = 0$. Set $B = V^{-T}\tilde{B}V^{-1}$ where ${V^{-1}}^T = V^{-T}, C = V^{-T}\tilde{C}V^{-1},\ A = V\tilde{A}V^{-1}$. Then

$$A^T B + BA + C = \left(V\tilde{A}V^{-1}\right)^T \left(V^{-T}\tilde{B}V^{-1}\right) + \left(V^{-T}\tilde{B}V^{-1}\right)\left(V\tilde{A}V^{-1}\right) + V^{-T}\tilde{C}V^{-1}$$

$$= V^{-T}\left(\tilde{A}^T B + \tilde{B}\tilde{A} + \tilde{C}\right)V^{-1} = 0$$

If $A^T B + BA + C = 0$ then set $\tilde{B} = V^T BV,\ \tilde{C} = V^T CV$ so $\tilde{A}\tilde{B} + \tilde{B}\tilde{A} + \tilde{C} = 0$.
For the 2×2 case, we consider 4 cases once again:

(1) $\Delta > 0$ then two real roots (eigenvalues). Jordan form of A is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Replace A by it's Jordan form. If $\lambda_1, \lambda_2 < 0$ then take $B = I$. $C = \begin{pmatrix} -2\lambda_1 & 0 \\ 0 & -2\lambda_2 \end{pmatrix}$ satisfies $A^T B + BA + C = 0$. If $\lambda_1, \lambda_2 \leq 0$ then can take any B, C.

(2) $\Delta < 0$ then has complex conjugate roots. $Av = v\lambda$ and $\overline{A}\overline{v} = \overline{v}\overline{\lambda}$. Take $\lambda = a + ib,\ v = x + iy$ with a, b, x, y real.

$$Ax = A\left(\frac{v + \overline{v}}{2}\right) = \frac{v\lambda + v\overline{\lambda}}{2} = xa - yb$$

$$Ay = A\left(\frac{v - \overline{v}}{2i}\right) = \frac{v\lambda - \overline{v}\overline{\lambda}}{2i} = xb + ya$$

So we have

$$A\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ \tilde{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Then $Re\lambda = Re\overline{\lambda} = a$. If $a < 0$ then $B = I$ and $C = \begin{pmatrix} -2a & 0 \\ 0 & -2a \end{pmatrix}$ satisfies $A^T B + BA + C = 0$, with B, C positive definite. If $a \leq 0$ then C is positive semi-definite.

(3) $\Delta = 0$ gives a repeated real root so A is similar to it's Jordan form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Take $B = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$.
If $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ then $A^T = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ so $A^T B = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda\mu \end{pmatrix}$, $BA = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda\mu \end{pmatrix}$.

$$A^T B + BA = \begin{pmatrix} 2\lambda & 1 \\ 1 & 2\lambda\mu \end{pmatrix} \text{ so } C = \begin{pmatrix} -2\lambda & -1 \\ -1 & -2\lambda\mu \end{pmatrix}$$

If $\lambda < 0$ then B, C are positive definite if we choose $\mu > 0$.

$$4\lambda^2\mu - 1 > 0 \ \Rightarrow \mu > \frac{1}{4\lambda^2}$$

If $\lambda \leq 0$ then there is nothing to prove.

(4) $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ then take $B = I$ and $C = -2A$. B is positive definite and C is positive definite if $\lambda < 0$ and positive semi-definite if $\lambda \leq 0$ for the 2×2 case.

□

The eigenvalue problem implies the existence of a solution to $A^T B + BA + C = 0$.
For linear systems, (strict)stability is equivalent to the existence of a (strict) Lyapunov function, which is equivalent to the eigenvalue condition.

5.5. Linearisation. Suppose ξ is an equilibrium of $x'(t) = F(x(t)),\ F(\xi) = 0$. Then, Taylor expanding,

$$F_j(x) = F_j(\xi) + \sum_{k=1}^{m} \frac{\partial F_j}{\partial x_k}(x_k - \xi_k) + remainder$$

$F_j(\xi) = 0$ as ξ is an equilibrium so set

$$A_{j,k} = \frac{\partial F_j}{\partial x_k}(\xi)\ y_k = (x_k - \xi_k)$$

so $F_j(x) = \sum_{k=1}^{m} A_{j,k} y_k + remainder$ and $y'(t) = x'(t) = F(x(t)) \approxeq Ay(t)$ and the error is small compared to $\|y\|$. We hope the stability properties of ξ as equilibria of $x' = F \circ x$ are the same as those of 0 as equilibrium of $y' = Ay$.

THEOREM 5.13. *If 0 is a stable equilibrium of the linearised equation then ξ is a strictly stable solution of the non-linear equation.*
Note: The converse is false and the theorem doesn't hold if we drop the word strict.

So for $y' = Ay$ and $y = x - \xi$ should approximately satisfy $y' = Ay$.
Warning: If the linearised system has a stable, but not strictly stable equilibrium at 0 then we learn nothing about ξ as an equilibrium of $x' = F \circ x$.
Example

$$u' = -v + \alpha(u^2 + v^2)u,\ v' = u + \alpha(u^2 + v^2)v \tag{11}$$

Then set $w = u^2 + v^2$ so $w' = 2uu' + 2vv' = 2\alpha(u^2 + v^2)^2 = 2\alpha w^2$. This is a separable equation, namely

$$\frac{dw}{dt} = 2\alpha w^2\ \Rightarrow \frac{dw}{w^2} = 2\alpha dt$$

Integrating this from $w(t_0) = w_0$ gives

$$\frac{1}{w_0} - \frac{1}{w} = 2(\alpha t - \alpha_0 t)$$

so

$$w = \frac{1}{\frac{1}{w_0} - 2(\alpha t - \alpha_0 t)} = \frac{w_0}{1 + 2\alpha t_0 w_0 - 2\alpha w_0 t}$$

If $\alpha = 0$ then $w = w_0$ (invariant) is stable, but not strictly stable.
If $\alpha > 0$ then $\lim_{t \to t_0 + \frac{1}{\alpha w_0}} w(t) = \infty$ and all non-equilibrium solutions blow up in finite time, so not stable.
If $\alpha < 0$ then $\lim_{t\to\infty} w(t) = 0$ so $\|(u,v) - (0,0)\| \to 0$. w is monotone decreasing so strictly stable. (11) has the same linearisation at $(0,0)$ independent of α. In this case $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

THEOREM 5.14. *With F, ξ, A as before, if $A^T B + BA + C = 0$ with B, C positive definite then $V(x) = (x - \xi)^T B(x - \xi)$ is a strict Lyapunov function for $x' = F \circ x$ at ξ.*

PROOF. It is clearly continuously differentiable with a strict local minimum at ξ. For monotonicity,

$$\frac{d}{dt}V(x(t)) = \left(\sum_{j=1}^{n} \frac{\partial V}{\partial x_j} F_j\right)(x(t))$$

and to evaluate this,

$$\frac{\partial V}{\partial x_j}(x) = 2\sum_{k=1}^{n} B_{j,k}(x_k = \xi_k)$$

and

$$F_j(x) = \sum_{l=1}^{n} A_{j,l}(x_l - \xi_l) + o(\|x - \xi\|)$$

using 'little o' notation. So then the derivative is equal to:

$$\left(\sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x) F_j\right)(x) = 2\sum_{j=1}^{n} A_{j,l} B_{j,k}(x_k - \xi_k)(x_l - \xi_l) + o(\|x - \xi\|^2)$$

$$= 2(x - \xi)^T (A^T B)(x - \xi) + o(\|x - \xi\|^2) = (x - \xi)^T (A^T B + BA)(x - \xi) + o(\|x - \xi\|^2)$$
$$-(x - \xi)^T C(x - \xi) + o(\|x - \xi\|^2) < 0$$

if $0 < \|x - \xi\| < \delta$. □

THEOREM 5.15. *If 0 is a strictly stable equilibrium of $y' = Ay$ then ξ is a strictly stable equilibrium of $x' = F \circ x$.*

Eigenvalues of A having negative real part implies the equilibrium is strictly stable.

THEOREM 5.16. *If the linearisation of $x' = F \circ x$ at ξ is unstable then ξ is an unstable equilibrium.*

We will not prove this.

Example Jacobi System.

$$x' = yz \; y' = -xz \; z' = -k^2 xy$$

Then we get

$$A = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & z & y \\ -z & 0 & -x \\ -k^2 y & -k^2 x & 0 \end{pmatrix}$$

At $(x, y, z) = (0, 0, 0)$, A is the zero matrix so linearisation is stable but not strictly stable so it provides no information about the system. At the other equilibria, the same thing happens.

Example Damped pendulum.

$$\theta' = \omega \; \omega' = -k \sin\theta - \lambda(\omega)\omega, \; \lambda \geq 0$$

then

$$A = \begin{pmatrix} \frac{\partial \theta'}{\partial \theta} & \frac{\partial \theta'}{\partial \omega} \\ \frac{\partial \omega'}{\partial \theta} & \frac{\partial \omega'}{\partial \omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k\cos\theta & -\lambda(\omega) - \omega\lambda'(\omega) \end{pmatrix}$$

and at $(\theta, \omega) = (n\pi, 0)$, this is equal to $\begin{pmatrix} 0 & 1 \\ (-1)^{n+1}k & -\lambda(0) \end{pmatrix}$. Then, $tr(A) = -\lambda(0)$ and $\det(A) = (-1)^n k$. If n is even and $\lambda(0) > 0$ then the linearisation is strictly stable, so equilibrium is strictly stable. If n odd then there is an eigenvalue with positive real part, so linearisation is unstable and the damped pendulum is unstable.

Example Predator Prey.

$$N' = PN(1 - \frac{N}{k}) - aNP \; P' = -cP + bNP$$

with all parameters positive. The equilibria are

$$(N, P) \in \{(0, 0), (k, 0), (\frac{c}{b} m \frac{r}{a}(1 - \frac{c}{bk}))\}$$

and then

$$A = \begin{pmatrix} \frac{\partial N'}{\partial N} & \frac{\partial N'}{\partial P} \\ \frac{\partial P'}{\partial N} & \frac{\partial P'}{\partial P} \end{pmatrix} = \begin{pmatrix} r - 2r\frac{N}{k} - ap & -aN \\ bNP & -c + bN \end{pmatrix}$$

At $(0, 0), A = \begin{pmatrix} r & 0 \\ 0 & -c \end{pmatrix}$ and the linearisation is unstable so $(0, 0)$ is an unstable equilibrium. At $(k, 0) A = \begin{pmatrix} -r & -ak \\ 0 & -c + bk \end{pmatrix}$ and we then have several cases. If $bk < c$ then the linearisation is strictly stable so $(k, 0)$ is strictly stable. If $bk > c$ then the linearisation is unstable so $(k, 0)$ is unstable.

Example Lorenz System.

$$x' = \sigma(y - x) \; y' = rx - y - xz \; z' = xy - bz$$

which has equilibria $(0, 0, 0)$ and $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r - 1)$ and once again

$$A = \begin{pmatrix} \frac{\partial x'}{\partial x} & \cdots & \\ \vdots & \ddots & \\ & & \frac{\partial z'}{\partial z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

The method we used previously only works for the 2×2 case, but in general we can use the Routh-Hurwitz algorithm to determine when the eigenvalues have negative real part.