AMM problems May 2014, due before 30 September

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

July 28, 2014

11775. *Proposed by Isaac Sofair.* Let $A_1, \ldots, A_k$ be finite sets. For $J \subseteq \{1, \ldots, k\}$, let $N_J = \left|\bigcup_{j \in J} A_j\right|$, and let $S_m = \sum_{J\colon |J|=m} N_J$.
(a) Express in terms of $S_1, \ldots, S_k$ the number of elements that belong to exactly m of the sets $A_1, \ldots, A_k$.
(b) Same question as in **(a)**, except that we now require the number of elements belonging to at least m of the sets $A_1, \ldots, A_k$.

11776. *Proposed by David Beckwith.* Given urns $U_1, \ldots, U_n$ in a line, and plenty of identical blue and identical red balls, let a_n be the number of ways to put balls into the urns subject to the conditions that
(i) Each urn contains at most one ball,
(ii) any urn containing a red ball is next to exactly one urn containing a blue ball, and
(iii) no two urns containing a blue ball are adjacent.
(a) Show that

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1+t+2t^2}{1-t-t^2-3t^3}.$$

(b) Show that

$$a_n = \sum_{j \geq 0} \sum_{m \geq 0} 4^j \left[\binom{n-2m}{j}\binom{m}{j} + \binom{n-2m-1}{j}\binom{m}{j} + 2\binom{n-2m}{j}\binom{m-1}{j} \right].$$

Here, $\binom{k}{l} = 0$ if $k < l$.

11777. *Proposed by Marian Dincă.* Let $x_1, \ldots, x_n$ be real numbers such that $\prod_{k=1}^{n} x_k = 1$. Prove that

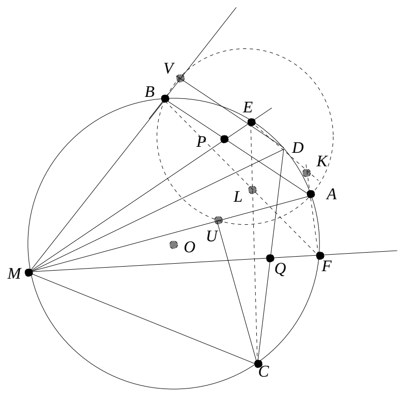
$$\sum_{k=1}^{n} \frac{x_k^2}{x_k^2 - 2x_k \cos(2\pi/n) + 1} \geq 1.$$

*This group involves students and staff of the Department of Mathematics, Trinity College, Dublin. Please address correspondence either to Timothy Murphy (`tim@maths.tcd.ie`), or Colm Ó Dúnlaing (`odunlain@maths.tcd.ie`).

11778. *Proposed by Li Zhou.* Let x, y, z be positive real numbers such that $x + y + z = \pi/2$. Let $f(x, y, z) = 1/(\tan^2 x + 4\tan^2 y + 9\tan^2 z)$. Prove that

$$f(x,y,z) + f(y,z,x) + f(z,x,y) \le \frac{9}{14}(\tan^2 x + \tan^2 y + \tan^2 z).$$

11779. *Proposed by Michel Bataille.* Let M, A, B, C, and D be distinct points (in any order) on a circle Γ with center O. Let the medians through M of triangles MAB and MCD cross lines AB and CD at P and Q, respectively, and meet Γ again at E and F, respectively. Let K be the intersection of AF with DE, and let L be the intersection of BF with CE. Let U and V be the orthogonal projections of C onto MA and D onto MB, respectively, and assume $U \neq A$ and $V \neq B$. Prove that A, B, U, and V are concyclic if and only if O, K, and L are collinear. **[The figure is inaccurate].**



11780. *Proposed by Cezar Lupu and Tudorel Lupu.* Let f be a positive-valued, concave function on $[0, 1]$. Prove that

$$\frac{3}{4}\left(\int_0^1 f(x)dx\right)^2 \le \frac{1}{8} + \int_0^1 f^3(x)dx.$$

11781. *Proposed by Roberto Tauraso.* For $n \ge 2$, call a positive integer *n-smooth* if none of its prime factors is larger than n. Let S_n be the set of all n-smooth positive integers. Let C be a finite, nonempty set of nonnegative integers, and let a and d be positive integers. Let M be the set of all positive integers of the form $m = \sum_{k=1}^{d} c_k s_k$, where $c_k \in C$ and $s_k \in S_n$ for $k = 1, \ldots, d$. Prove that there are infinitely many primes p such that $p^a \notin M$.