AMM problems August-September2013, due before 30 November 2013

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

October 30, 2013

11719. *Proposed by Nicolae Anghel.* Let f be a twice-differentiable function from $[0, \infty)$ into $(0, \infty)$ such that

$$\lim_{x\to\infty} \frac{f''(x)}{f(x)(1+f'(x)^2)^2} = \infty.$$

Show that

$$\lim_{x\to\infty} \int_{t=0}^{x} \frac{\sqrt{1+f'(t)^2}}{f(t)} dt \int_{t=x}^{\infty} \sqrt{1+f'(t)^2} f(t) dt = 0.$$

11720. *Proposed by Ira Gessel.* Let $E_n(t)$ be the Eulerian polynomial defined by

$$\sum_{k=0}^{\infty} (k+1)^n t^k = \frac{E_n(t)}{(1-t)^{n+1}},$$

and let B_n be the nth Bernoulli number. Show that $(E_{n+1}(t) - (1-t)^n)B_n$ is a polynomial with integer coefficients.

11721. *Proposed by Roberto Tauraso.* Let p be a prime greater than 3, and let q be a complex number other than 1 such that $q^p = 1$. Evaluate

$$\sum_{k=1}^{p-1} \frac{(1-q^k)^5}{(1-q^{2k})^3(1-q^{3k})^2}.$$

11722. *Proposed by Nguyen Thanh Binh.* Let ABC be an acute triangle in the plane, and let M be a point inside ABC. Let $O_1, O_2,$ and O_3 be the circumcenters of BCM, CAM, and ABM, respectively. Let c be the circumcircle of ABC. Let D, E, and F be the points opposite A, B, and C, respectively, at which AM, BM, and CM meet c. Prove that O_1D, O_2E, and O_3F are concurrent at

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a point P that lies on c.

11723. *Proposed by L.R. King.* Let A, B, and C be three points in the plane, and let D, E, and F be points lying on BC, CA, AB, respectively. Show that there exists a conic tangent to BC, CA, and AB at D, E, and F, respectively, if and only if AD, BE, and CF are concurrent.

11724. *Proposed by Andrew Cusumano.* Let $f(n) = \sum_{k=1}^{n} k^k$ and let $g(n) = \sum_{k=1}^{n} f(k)$. Find

$$\lim_{n\to\infty} \frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)}.$$

11725. *Proposed by Mher Safaryan.* Let m be a positive integer. Show that, as $n \to \infty$,

$$\left|\log 2 - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\right| = \frac{C_1}{n} + \frac{C_2}{n^2} + \ldots + \frac{C_m}{n^m} + o\left(\frac{1}{n^m}\right)$$

where

$$C_k = (-1)^k \sum_{i=1}^{k} \frac{1}{2^i} \sum_{j=1}^{i} (-1)^j \binom{i-1}{j-1} j^{k-1}$$

for $1 \le k \le m$.