1 AMM problem 11651

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Show that

$$\lfloor \frac{n+1}{\phi} \rfloor = n - \lfloor \frac{n}{\phi} \rfloor + \lfloor \frac{\lfloor n/\phi \rfloor}{\phi} \rfloor - \lfloor \frac{\lfloor \frac{\lfloor n/\phi \rfloor}{\phi} \rfloor}{\phi} \rfloor \ldots \tag{1.1}$$

holds for every nonnegative integer n if and only if $\phi = (1+\sqrt{5})/2$.

Answer. The right-hand side may be written as $E(n)$, and clearly $E(n) = n - E(\lfloor n/\phi \rfloor)$. This will not converge unless $\phi > 1$. We assume from now on that $\phi > 1$.

If (1.1) holds for all n, then

$$\lfloor \frac{n+1}{\phi} \rfloor + \lfloor \frac{\lfloor n/\phi \rfloor + 1}{\phi} \rfloor = n \tag{1.2}$$

On the other hand, if (1.2) holds for all n then

$$\lfloor \frac{n+1}{\phi} \rfloor = n - \lfloor \frac{\lfloor n/\phi \rfloor + 1}{\phi} \rfloor$$

$$\lfloor \frac{\lfloor n/\phi \rfloor + 1}{\phi} \rfloor = \lfloor \frac{n}{\phi} \rfloor - \lfloor \frac{\lfloor \frac{\lfloor \frac{n}{\phi} \rfloor}{\phi} \rfloor + 1}{\phi} \rfloor \ldots$$

whence the identity (1.1) can be 'unrolled.' We discard the original identity in favour of the equivalent (1.2).

The latter identity implies

$$\frac{n}{\phi} + \frac{n}{\phi^2} = n + O(1)$$

for all n. Dividing by n,

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1 + O(1/n)$$

for arbitrarily large n, which is only possible, since $\phi > 1$, if ϕ is the golden section $(1+\sqrt{5})/2$.

To deal with the converse, we assume that ϕ is indeed the golden section. We reserve ψ to denote the other root of $x^2 - x - 1 = 0$, i.e., $\psi = (1-\sqrt{5})/2 = -1/\phi$.
Let $m = \lfloor n/\phi \rfloor$. We need to prove

$$\lfloor \frac{n+1}{\phi} \rfloor + \lfloor \frac{m+1}{\phi} \rfloor = n$$

for every nonnegative integer n. Write

$$\frac{n}{\phi} = m + \alpha, \quad \text{so} \quad \frac{n+1}{\phi} = m + \alpha + \frac{1}{\phi}.$$

Since $n/\phi + n/\phi^2 = n$,

$$m + \alpha + \frac{n}{\phi^2} = n$$

Case (i): $\alpha + 1/\phi < 1$, in which case $m = \lfloor (n+1)/\phi \rfloor$, and it is enough to show that

$$n - m = \alpha + \frac{n}{\phi^2} = \lfloor \frac{m+1}{\phi} \rfloor.$$

or

$$\alpha + \frac{n}{\phi^2} < \frac{m+1}{\phi} < \alpha + \frac{n}{\phi^2} + 1.$$

Since $m/\phi < n/\phi^2$, the second inequality is obvious. The first is equivalent to

$$\frac{n}{\phi^2} + \alpha < \frac{n/\phi + 1 - \alpha}{\phi}$$

or

$$\alpha < \frac{1-\alpha}{\phi}$$

But $\alpha < 1 - 1/\phi = 1/\phi^2$ and $1 - \alpha > 1 - 1/\phi^2 = 1/\phi$, so this is correct.

Case (ii): $\alpha + 1/\phi > 1$. Then $\lfloor (n+1)/\phi \rfloor = m + 1$, and we need to show

$$m + 1 + \lfloor \frac{m+1}{\phi} \rfloor - \frac{n}{\phi} - \frac{n}{\phi^2} = 0.$$

So we need to show that

$$\frac{n}{\phi} - \alpha + 1 + \frac{n/\phi - \alpha + 1}{\phi} - \frac{n}{\phi} - \frac{n}{\phi^2}$$
$$= 1 - \alpha + \frac{1-\alpha}{\phi} = (1-\alpha)(1 + 1/\phi) = (1-\alpha)\phi$$

is between 0 and 1. It is positive, and $\alpha > 1 - 1/\phi = 1/\phi^2$, so $1 - \alpha < 1 - \phi^2 = 1/\phi$, as required. ∎