

## Problem solving methods II: divisibility and combinatorics (15/01/2013)

1. Prove that there is no group of 2009 people where each person knows exactly 1981 other persons from this group.
2. How many ways are there to elect the secretary, the treasurer, and the librarian from the 100 members of a student society? How many ways are there to elect three executive officers from the 100 society members? Why are the answers to these two questions different?
3. How many ways are there to label the faces of a cube by integers  $1, 2, \dots, 6$ ? Every integer should be used, two labellings are the same if they can be identified rotating the cube.
4. Show that the product of every three consecutive integers is (a) divisible by 3; (b) divisible by 6. (c) Show that the product of every four consecutive integers is divisible by 24. (d) Show that the product of every  $k$  consecutive integers is divisible by  $k!$ .
5. How many permutations  $\sigma$  of  $1, 2, \dots, n$  move each number from its place (that is,  $\sigma(i) \neq i$  for all  $i$ )?
6. Compute the Euler function  $\phi(n)$ , that is the number of integers between 0 and  $n$  that are coprime to  $n$ , in terms of prime factor decomposition of  $n$ .
7. (a) In a lab,  $2n$  students are present for a class, and the teacher needs to pair them up for performing a certain experiment. How many ways are there to do that? (b) Show that  $\frac{(2n)!}{2^n \cdot n!}$  is an integer. (c) Show that  $\frac{(kn)!}{k^n \cdot n!}$  is an integer.
8. (a) There are  $n$  petrol stations on a circular road, positioned in some way, not necessarily at same distances; also they all have different amounts of petrol, but altogether they have enough petrol to travel the full circle. Show that it is possible for a car to start from one of the stations, re-fuelling along the way, so that it can travel the full circle just using the petrol from stations along the way. (b) For a sequence  $a_1, \dots, a_{2n+1}$  of length  $2n + 1$ ,  $n$  entries are equal to  $-1$  and  $n + 1$  remaining entries are equal to 1. Show that there is exactly one cyclic shift  $a_k, \dots, a_{2n+1}, a_1, \dots, a_{k-1}$  of this sequence for which all the partial sums (the first term, the first two terms etc.) are non-negative. (c) Show that  $\frac{(2n)!}{n!(n+1)!}$  is an integer.
9. (a) A convex polygon with  $n$  vertices is dissected into triangles by its diagonals. Prove that the number of triangles is equal to  $n - 2$  and is independent of the choice of diagonals that dissect the polygon. (b\*) How many ways are there to perform such a dissection? (To make the set-up clear: there are 2 ways for a quadrilateral, 5 ways for a pentagon, etc.)
10. For a set of 2013 coins, it is known that if we remove an arbitrary coin from it, the remaining ones can be divided into two sets of 1006 coins of the same total weight. Show that all coins have the same weight. (Maybe it would help to first assume that all weights are integers.)
11. (a) A rectangle is cut into smaller rectangles, and for each of those the length of one of the sides is divisible by 5. Show that the length of one of the sides of the original rectangle is divisible by 5. Assume that all sides of all rectangles sides have integer lengths. (b) A rectangle is cut into smaller rectangles, and for each of those the length of one of the sides is an integer. Show that the length of one of the sides of the original rectangle is an integer.

Some catchphrases:

- Two basic operations that make combinatorial meaning are addition and multiplication. Addition is needed when dealing with disjoint unions, multiplication — when dealing with Cartesian products of sets. Division happens usually when you count things with repetitions, and each thing is counted the same number of times.
- In general, unions are more tricky, but are controlled by the inclusion–exclusion formula, one of the most important results in elementary combinatorics.
- When proving that some fraction is an integer, it could be useful to interpret it as a number of ways of doing something.
- When dealing with real numbers, it sometimes helps to reduce a problem to a similar problem about integers, and then use that result, directly or indirectly. Also it may work the other way round. In any case, do assume that various areas of maths are interrelated (because they really are).