# Solutions to MA3417 Homework assignment 4

1. (i) In the case $a = b = c = 0$, we are essentially dealing with the ideal $I = (y^2)$, after dividing $f$ by $d \neq 0$. Computing dimensions for small $n = 0, 1, 2, 3, 4$, we get $1, 2, 3, 5, 8$, so one suspects Fibonacci numbers. This is easy to prove by induction: a normal monomial of degree $n$ either starts with $x$, in which case the rest can be an arbitrary normal monomial of degree $n-1$, or starts with $y$, in which case it must be followed by $x$ which is then followed by an arbitrary normal monomial of degree $n-2$, so $d_n = d_{n-1} + d_{n-2}$, and the claim follows.

   (ii) In the case $a = b = 0$, $c \neq 0$, we are essentially dealing with an ideal $I = (yx + dy^2)$, after dividing $f$ by $c \neq 0$. For the `glex` ordering with $x > y$, the leading monomial is $yx$, which does not have nontrivial small common multiples with itself, so $yx + dy^2$ forms the reduced Gröbner basis. The normal monomials are $x^i y^j$, so $d_n = n + 1$.

   (iii) In the case $a = 0$, $b \neq 0$, , we are essentially dealing with an ideal $I = (xy + cyx + dy^2)$, after dividing $f$ by $b \neq 0$. For the `glex` ordering with $x > y$, the leading monomial is $xy$, which does not have nontrivial small common multiples with itself, so $xy + cyx + dy^2$ forms the reduced Gröbner basis. The normal monomials are $y^i x^j$, so $d_n = n + 1$.

2. The leading monomial of $x^2 + bxy + cyx + dy^2$ is $x^2$, which already ensures that the normal monomials of degree 3 are among $xyx$, $xy^2$, $yxy$, $y^2x$ and $y^3$. Since the cosets of normal monomials form a basis in the quotient, the dimension $d_3$ does not exceed 5. Also, there may be at most one new constraint, arising from the only S-polynomial of degree 3 that is there, the one corresponding to the small common multiple $x^3$ of the leading term of $f$ with itself.

   Let us compute that S-polynomial. It is

   $$(x^2+bxy+cyx+dy^2)x-x(x^2+bxy+cyx+dy^2) = (b-c)xyx+cyx^2+dy^2x-bx^2y-dxy^2.$$

   The remainder of this after long division by $x^2 + bxy + cyx + dy^2$ is, by a direct computation,

   $$(b-c)xyx + (b^2-d)xy^2 + (d-c^2)y^2x + (b-c)dy^3.$$

   If this is equal to zero, $d_3 = 5$. Otherwise, we add one linear dependence between cosets of normal monomials, and $d_3 = 4$.

   Note that the remainder is zero whenever the equations $b = c$, $b^2 = d$, $c^2 = d$, $(b-c)d = 0$ are satisfied. All of those follow from $b = c$, $d = b^2$. Thus, for $(b, c, d) = (b, b, b^2)$ we have $d_3 = 5$ and otherwise $d_3 = 4$.

3. In the case $d_3 = 5$, $f$ forms a reduced Gröbner basis, so $d_n$ is equal to the number of monomials of degree $n$ not divisible by $x^2$, which is equal to a Fibonacci number by Problem 1.

4. Suppose $d_3 = 4$. This means that the remainder

$$(b-c)xyx + (b^2-d)xy^2 + (d-c^2)y^2x + (b-c)dy^3$$

computed in Problem 2 is non-zero. Let us consider two cases.

Case 1: $b \neq c$. In this case, $xyx$ is the leading monomial of that remainder. The only small common multiples of degree 4 are $xyx^2$ and $x^2yx$; all others give rise to S-polynomials of higher degrees, and this will not affect $d_4$. The corresponding S-polynomials are

$$(x^2+bxy+cyx+dy^2)yx - \frac{1}{b-c}x((b-c)xyx+(b^2-d)xy^2+(d-c^2)y^2x+(b-c)dy^3)$$
$$= bxy^2x + cyxyx + dy^3x - \frac{b^2-d}{b-c}x^2y^2 - \frac{d-c^2}{b-c}xy^2x - dxy^3$$

and

$$\frac{1}{b-c}((b-c)xyx+(b^2-d)xy^2+(d-c^2)y^2x+(b-c)dy^3)x - xy(x^2+bxy+cyx+dy^2)$$
$$= \frac{b^2-d}{b-c}xy^2x + \frac{d-c^2}{b-c}y^2x^2 + dy^3x - bxyxy - cxy^2x - dxy^3,$$

and their remainders after long division by $x^2+bxy+cyx+dy^2$ and $xyx + \frac{b^2-d}{b-c}xy^2 + \frac{d-c^2}{b-c}y^2x + dy^3$ are both equal to

$$(b^2-bc+c^2-d)xy^2x + (b^3-2bd+cd)xy^3 + (bd+c^3-2cd)y^3x + d(b^2-bc+c^2-d)y^4.$$

This vanishes if and only if all the coefficients $b^2-bc+c^2-d$, $b^3-2bd+cd$, $bd+c^3-2cd$, and $d(b^2-bc+c^2-d)$ vanish. Note that if $b^2-bc+c^2-d=0$ and $b^3-2bd+cd=0$, then

$$0 = b^3 - 2b(b^2-bc+c^2) + c(b^2-bc+c^2) = c^3 - 3c^2b + 3cb^2 - b^3 = (c-b)^3.$$

Since we assume $b \neq c$, this case is impossible, so there is a nonzero remainder in this case. This implies that the leading monomials of degree up to 4 of the reduced Gröbner basis are $x^2$, $xyx$, and one element of degree 4 that is normal with respect to $x^2$ and $xyx$. The number of normal monomials of degree 4 with respect to $x^2$ is equal to 8, $xyx$ prohibits $xyxy$ and $yxyx$ among those, and one extra monomial prohibits one more element, thus we get $d_4 = 5$ in this case.

Case 2: $b = c$, but $b^2 \neq d$. In this case, the remainder of the S-polynomial in degree 3 is, in self-reduced form, $xy^2 - y^2x$. By a direct computation, the S-polynomial corresponding to the only new common multiple $x^2y^2$ of the leading terms has zero remainder, so these two polynomials form the reduced Gröbner basis. The number of normal monomials of degree 4 with respect to $x^2$ is equal to 8, $xy^2$ prohibits $yxy^2$, $xy^2x$ and $xy^3$ among those, so we get $d_4 = 5$ in this case.