**MA341D Answers and solutions to homework assignment 2**

1. The Magma commands

```
Q := RationalField();
P<x,y,z> := PolynomialRing(Q, 3, "lex");
S:=[x*y-z^2-z,y*z-x^2-x,x*z-y^2+y];
GroebnerBasis(S);
```

produce the following Gröbner basis:

$$x^2 + x - z^2 - z,$$
$$xy - z^2 - z,$$
$$xz,$$
$$y^2 - y,$$
$$yz - z^2 - z,$$
$$z^3 + z^2.$$

From the last equation, $z = 0$ or $z = -1$. If $z = 0$, the equations become $x^2 + x = 0$, $xy = 0$, $y^2 - y = 0$, leading to the solutions $(0, 0, 0)$, $(-1, 0, 0)$, and $(0, 1, 0)$. If $z = -1$, the equations become $x^2 + x = 0$, $xy = 0$, $-x = 0$, $y^2 - y = 0$, $-y = 0$, leading to the solution $(0, 0, -1)$. Those four solutions together form the complete solution set.

2. The Magma commands

```
Q := RationalField();
P<x,y,z> := PolynomialRing(Q, 3, "lex");
S:=[x*y-z^2-z,y*z-x^2-x,x*z-y^2-y];
GroebnerBasis(S);
```

produce the following Gröbner basis:

$$x^2 + x - yz,$$
$$xy - z^2 - z,$$
$$xz + yz + z^2 + z,$$
$$y^2 + yz + y + z^2 + z.$$

This means that the elimination ideal $I_2$ is $\{0\}$. Using the Extension Theorem, we conclude that since there is a polynomial in our Gröbner basis with the leading term $y^2$, every $z$ can be extended to a solution $(y, z)$ to the elimination ideal $I_1 = (y^2 + yz + y + z^2 + z)$. Moreover, since the discriminant of $y^2+yz+y+z^2+z$ as a polynomial in $y$ is $(1+z)(1-3z)$, for each value of $z$ except for $-1$ and $1/3$ we can find two distinct values of $y$, for $z = -1$ we have $y = 0$, and for $z = 1/3$ we have $y = -2/3$. Furthermore, since there is a polynomial with the leading term $x^2$, every solution $(y, z)$ to $I_1$ extends to a solution $(x, y, z)$. If

$z \neq 0$, the third equation shows that there is only one solution $x = -(y+z+1)$. If $z = 0$, we should look at common roots of the polynomials become

$$x^2 + x,$$
$$xy,$$
$$y^2 + y.$$

which are $(0,0)$, $(-1,0)$ and $(0,-1)$. Altogether the solution set can be described as

$$\{(0,0,0), (-1,0,0), (0,-1,0), (-y-z-1, y, z) \colon y^2 + yz + y + z^2 + z = 0, z \neq 0\},$$

or if we note that the second and the third point are precisely the values of the third point for $z = 0$,

$$\{(0,0,0), (-y-z-1, y, z) \colon y^2 + yz + y + z^2 + z = 0\}.$$

3. (a) We introduce two new variables $a$ and $b$, and look for the extremal points of the function

$$F(x,y,z,a,b) = (x^3 + y^3 + z^3) - a(x+y+z) - b(x^2 + y^2 + z^2 - 1/2).$$

Those extremal points are common zeros of $\partial_x F = 3x^2 - 2bx - a$, $\partial_y F = 3y^2 - 2by - a$, $\partial_z F = 3z^2 - 2bz - a$, $\partial_a F = -(x+y+z)$, $\partial_b F = -(x^2 + y^2 + z^2 - 1/2)$.

(b) The `Magma` commands

```
Q := RationalField();
P<a, b, x, y, z> := PolynomialRing(Q, 5, "lex");
S := [
x+y+z,
x^2+y^2+z^2-1/2,
3*x^2-b*2*x-a,
3*y^2-b*2*y-a,
3*z^2-b*2*z-a
];
GroebnerBasis(S);
```

produce the following Gröbner basis:

$$a - 1/2,$$
$$b - 9z^3 + 9/4z,$$
$$x + y + z,$$
$$y^2 + yz + z^2 - 1/4,$$
$$yz^2 - 1/12y + 1/2z^3 - 1/24z,$$
$$z^4 - 5/12z^2 + 1/36.$$

(c) Factorizing the last equation, we get $(z^2 - 1/3)(z^2 - 1/12) = 0$. Let us consider those two cases individually.

Suppose $z^2 - 1/12 = 0$. Adding to our lit of polynomials $z^2 - 1/12$ and recomputing the Gröbner basis, we get

$$a - 1/2,$$
$$b + 3/2z,$$
$$x + y + z,$$
$$y^2 + yz - 1/6,$$
$$z^2 - 1/12$$

From the last equation, $z = \pm\frac{1}{2\sqrt{3}}$. Thus, we have $0 = y^2 \pm \frac{1}{2\sqrt{3}}y - 1/6 = (y \pm \frac{1}{\sqrt{3}})(y \mp \frac{1}{2\sqrt{3}})$, so the partial solutions $(y, z)$ are

$$(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}),$$

and from $x + y + z = 0$ each of those extends uniquely to a solution, obtaining

$$(-\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}), (\frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}), (-\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}),$$

Suppose $z^2 - 1/3 = 0$. Adding to our lit of polynomials $z^2 - 1/12$ and recomputing the Gröbner basis, we get

$$a - 1/2,$$
$$b - 3/4z,$$
$$x + 1/2z,$$
$$y + 1/2z,$$
$$z^2 - 1/3$$

From the last equation, $z = \pm\frac{1}{\sqrt{3}}$. Substituting that into the previous ones, we obtain two more solutions

$$(-\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}).$$

4. (a) Note that $x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_i x_j^{k-2} + x_j^{k-1} = 0$ if and only if $x_i^k = x_j^k$ and $x_i \neq x_j$. Also, $x_i^k = 1$ for all $k$, so effectively our polynomials have a common zero if and only if they have a common zero where every coordinate is a $k$-th root of unity and those roots at positions $i$ and $j$ are different if and only if the vertices $i$ and $j$ are connected with an edge. This is precisely the regular colouring condition.

   (b) Let us denote those vertices by $a, b, c, d, e, f, g, h$ clockwise starting from the top one. Then the corresponding polynomials are

$$a^3 - 1, b^3 - 1, c^3 - 1, d^3 - 1, e^3 - 1, f^3 - 1, g^3 - 1, h^3 - 1,$$
$$a^2 + ac + c^2, a^2 + af + f^2, a^2 + ag + g^2,$$
$$b^2 + bc + c^2, b^2 + be + e^2, b^2 + bg + g^2,$$
$$b^2 + bh + h^2, c^2 + cd + d^2, c^2 + ch + h^2,$$
$$d^2 + de + e^2, d^2 + dh + h^2, e^2 + ef + f^2,$$
$$e^2 + eg + g^2, f^2 + fg + g^2.$$

   The `Magma` commands

```
Q := RationalField();
P<a,b,c,d,e,f,g,h> := PolynomialRing(Q, 8, "lex");
S := [
a^3-1, b^3-1, c^3-1, d^3-1, e^3-1, f^3-1, g^3-1,h^3-1,
a^2+a*c+c^2, a^2+a*f+f^2, a^2+a*g+g^2,
b^2+b*c+c^2, b^2+b*e+e^2, b^2+b*g+g^2, b^2+b*h+h^2,
c^2+c*d+d^2, c^2+c*h+h^2,
d^2+d*e+e^2, d^2+d*h+h^2,
e^2+e*f+f^2, e^2+e*g+g^2,
f^2+f*g+g^2
];
GroebnerBasis(S);
```

output the result

$$\begin{gathered} a-h, \\ b+g+h, \\ c-g, \\ d+g+h, \\ e-h, \\ f+g+h, \\ g^2+gh+h^2, \\ h^3-1, \end{gathered}$$

which mean that there exists a regular colouring (since otherwise the reduced Gröbner basis would consist of just 1), and that if we choose a colour of the vertex $h$, then the vertex $g$ has two possible choices of colour, and colours of other vertices are reconstructed uniquely: $f$ is the third colour different from $g$ and $h$, $c$ is the same as $g$, $a$ and $e$ the same as $h$, and both $b$ and $d$ the same as $f$. Altogether, there are 6 different colourings.