Lattice Paths and Determinants

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1 Introduction

We present the proof of an interesting result relating matrix determinants and graphs, and its application both as a tool to prove other identities and to relate certain determinant problems to subject of counting paths in a lattice, following [1] and the talk by Oisin Flynn-Connolly.

2 The Lemma

Let $M = (m_{ij})_{1 \leq i,j \leq n}$ be a real $n \times n$ matrix. We have

$$\det M = \sum_{\sigma \in S_n} m_{1\sigma(1)} \cdots m_{n\sigma(n)}$$

Consider a weighted directed bipartite graph with vertex sets $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$, and edges from each $A_i \rightarrow B_j$ weighted by $m_{ij}$.

Definition 2.1. The weight $w(P)$ of a path $P$ in a weighted directed graph is the product $\prod_{e \in P} w(e)$ of the weights of the edges in the path.

Definition 2.2. If $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_n\}$ are vertices in a directed graph, then a path system $\mathcal{P}$ from $A$ to $B$ is a collection of paths $P_i : A_i \rightarrow B_{\sigma(i)}$ for some $\sigma \in S_n$. We define $\operatorname{sgn}(\mathcal{P}) = \operatorname{sgn}(\sigma)$ and the weight of $\mathcal{P}$

$$w(\mathcal{P}) = w(P_1)w(P_2)\cdots w(P_n)$$

For the graph defined above, we have

$$\det M = \sum_{\mathcal{P} : A \rightarrow B} \operatorname{sgn}(\mathcal{P})w(\mathcal{P})$$

It turns out that this is a special case of a useful general result, originally proven by Ernst Lindström in 1972, and rediscovered in 1985 by Ira Gessel and Gerard Viennot to apply to various combinatorial problems.

Let $G = (V, E)$ be a finite weighted acyclic directed graph. Let $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_n\}$ be subsets of $V$, not necessarily disjoint.

Definition 2.3. The path matrix from $A$ to $B$ is the matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ with $m_{ij} = \sum_{P : A_i \rightarrow B_j} w(P)$ (summation over all paths).

A path system $\mathcal{P} = (P_1, \ldots, P_n)$ is said to be vertex-disjoint if the paths $P_i$ are pairwise vertex-disjoint, i.e., have no vertices in common.

Lemma 2.1 (Lindström, Gessel, Viennot). With $A, B, \mathcal{P}, M$ as before we have

$$\det(M) = \sum_{\mathcal{P} : A \rightarrow B \text{ vertex-disjoint}} \operatorname{sgn}(\mathcal{P})w(\mathcal{P})$$
Proof. By grouping the path systems $P_\sigma$ corresponding to different $\sigma$, we find

$$\det(M) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \sum_{P_i : A_i \to B_{\sigma(i)}} w(P_i) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{P_\sigma : A \to B} w(P_\sigma) = \sum_{\mathcal{P} : A \to B} \text{sgn}(\mathcal{P}) w(\mathcal{P})$$

with summation over all path systems.

We therefore must show

$$\sum_{\mathcal{P} : A \to B \text{ not v-d}} \text{sgn}(\mathcal{P}) w(\mathcal{P}) = 0$$

Let $N$ be the set of non-vertex-disjoint path systems. We shall define a bijective map $\pi : N \to N$ satisfying $w(\pi(\mathcal{P})) = w(\mathcal{P})$ and $\text{sgn}(\pi(\mathcal{P})) = -\text{sgn}(\mathcal{P})$. Let $\mathcal{P} = (P_1, \ldots, P_n)$ be non-vertex-disjoint. Let $i_0$ be the minimal $i$ such that $P_i$ shares a vertex with some other $P_j$. Let $Q$ be the first vertex in the path $P_{i_0}$ that is shared with some $P_j$. Let $j_0 > i_0$ be the minimal $j \neq i$ such that $P_j$ passes through $Q$. Thus we have paths

$$X : A_{i_0} \to Q$$
$$Y : A_{j_0} \to Q$$
$$Z : Q \to B_{\sigma(i_0)}$$
$$W : Q \to B_{\sigma(j_0)}$$

with $P_{i_0} = Z \cdot X$ and $P_{j_0} = W \cdot Y$. Let $P'_{i_0} = W \cdot X : A_{i_0} \to B_{\sigma(j_0)}$ and $P'_{j_0} = Z \cdot Y : A_{j_0} \to B_{\sigma(i_0)}$. Then if $P'_k = P_k$ for $i_0 \neq k \neq j_0$, we may define $\pi(\mathcal{P}) = (P'_1, \ldots, P'_n)$. This satisfies $w(\pi(\mathcal{P})) = w(\mathcal{P})$, as $w(P'_{i_0}) w(P'_{j_0}) = w(X) w(Y) w(Z) w(W) = w(P_{i_0}) w(P_{j_0})$ and all other factors are the same. We have $P'_{i_0} : A_k \to B_{\sigma'(k)}$, where $\sigma'(i_0) = \sigma(j_0)$, $\sigma'(j_0) = \sigma(i_0)$ and $\sigma'(k) = \sigma(k)$ for $i_0 \neq k \neq j_0$. Thus $\sigma'$ differs from $\sigma$ by a transposition and $\text{sgn}(\sigma') = -\text{sgn}(\sigma)$. So

$$\text{sgn}(\pi(\mathcal{P})) = \text{sgn}(\sigma') = -\text{sgn}(\sigma) = -\text{sgn}(\mathcal{P})$$

as required. $\pi$ is bijective as we see from its construction that $\pi(\pi(\mathcal{P})) = \mathcal{P}$, as $i_0, Q, j_0$ will be the same for $\pi(\mathcal{P})$ as $\mathcal{P}$

Thus

$$\sum_{\mathcal{P} : A \to B \text{ not v-d}} \text{sgn}(\mathcal{P}) w(\mathcal{P}) = \sum_{\mathcal{P} : A \to B \text{ not v-d}} \text{sgn}(\pi(\mathcal{P})) w(\pi(\mathcal{P})) = -\sum_{\mathcal{P} : A \to B \text{ not v-d}} \text{sgn}(\mathcal{P}) w(\mathcal{P})$$

$$\implies \sum_{\mathcal{P} : A \to B \text{ not v-d}} \text{sgn}(\mathcal{P}) w(\mathcal{P}) = 0$$

$$\therefore \sum_{\mathcal{P} : A \to B \text{ vertex-disjoint}} \text{sgn}(\mathcal{P}) w(\mathcal{P}) = \sum_{\mathcal{P} : A \to B} \text{sgn}(\mathcal{P}) w(\mathcal{P})$$

$\square$

3 Further Results

This result has various useful applications. Firstly, we may use it to prove the following formula for the determinant of a product of non-square matrices.
Theorem 3.1. If \( P = (p_{ij}) \) is an \( r \times s \) matrix and \( Q = (q_{ij}) \) is an \( s \times r \) matrix, \( r \leq s \), then

\[
\det(PQ) = \sum_{Z} \det(P_Z) \det(Q_Z)
\]

with summation over subsets \( Z \subset \{1, \ldots, s\} \) of size \( r \), \( P_Z \) the \( r \times r \) matrix with column-set \( Z \) and \( Q_Z \) the \( r \times r \) matrix with row-set \( Z \).

Proof. We construct a weighted directed bipartite graph between \( A = \{A_1, \ldots, A_r\} \) and \( B = \{B_1, \ldots, B_s\} \) with edges \( A_i \to B_j \) having weights \( p_{ij} \). We construct also a graph between \( B \) and \( C = \{C_1, \ldots, C_s\} \) with edges \( B_i \to C_j \) having weights \( q_{ij} \). When put together these give a graph \( G \) between \( A \) and \( C \) with path matrix \( M = PQ \):

\[
m_{ij} = \sum_{P:A_i \to C_j} w(P) = \sum_{k=1}^{s} p_{ik}q_{kj} = (PQ)_{ij}
\]

since paths \( A_i \to C_j \) pass through some \( B_k \).

There are many other applications of the lemma to specific problems. For example, one might ask the following question:

Given some integers \( a_1 < \cdots < a_n, b_1 < \cdots < b_n \) what is the determinant of the matrix with entries \( m_{ij} = \binom{a_i}{b_j} \)? (if \( a < b \) then \( \binom{a}{b} = 0 \))

Let us consider an \( a \times b \) lattice of points. How many ways are there to travel from the bottom left corner to the top right corner by a series of steps north and east between points? We may represent such a journey as a string made up of the letters \( N \) and \( E \), and in order to travel from one corner to the other, the string must contain \( a \) Es and \( b \) Ns. The number of such strings is the number of ways to choose \( b \) positions in a string of length \( a + b \), i.e. \( \binom{a+b}{b} \).

Returning to the original problem, construct a graph whose vertices are integer points in the region of the \( xy \) plane between \( x = 0 \) and \( y = -x \) with \( y \) negative (cut off at some sufficiently low level), and whose edges point north or east between adjacent vertices all with weight one as shown below. Let \( A_i = (0, -a_i) \) and \( B_i = (b_i, -b_i) \). Then the graph between \( A_i \) and \( B_j \) forms an \( (a_i - b_j) \times b_j \) lattice, and the path matrix \( M \) between \( A \) and \( B \) is

\[
m_{ij} = \sum_{P:A_i \to B_j} 1 = \binom{(a_i - b_j) + b_j}{b_j} = \binom{a_i}{b_j}
\]
Vertex-disjoint path systems must go $A_i \rightarrow B_i$ (i.e. $\sigma = 1$) otherwise the paths would cross. Thus $\det(M)$ is the number of lattice path systems $(P_i : A_i \rightarrow B_i)_{i=1}^n$, so it is always positive and in particular $\det(M) = 0 \iff \exists i : a_i < b_i$.

References