A TERRIBLE WAY TO APPROXIMATE π

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1. Introduction

In 1901, a mathematician by the name of Lazzarini performed a remarkable experiment; by dropping a small stick on a ruled piece of paper 3408 times, he was able to estimate π to six decimal places! The experiment was remarkable for two reasons - the first of these was his method. What could the configuration of a few needles on a piece of paper have to do with π? It turns out that the mathematics behind this problem is much older than Lazzarini - it goes back to a problem posed by Le Compte de Buffon in 1777:

Question (Buffon's Needle Problem)**.** *Suppose drop a needle of length l onto a piece of paper on which has been drawn evenly spaced parallel lines of distance d apart, as in figure 1. Suppose that this needle is shorter than the distance between the lines - i.e. $d \geq l$. What is the probability that the needle crosses one of the lines?*

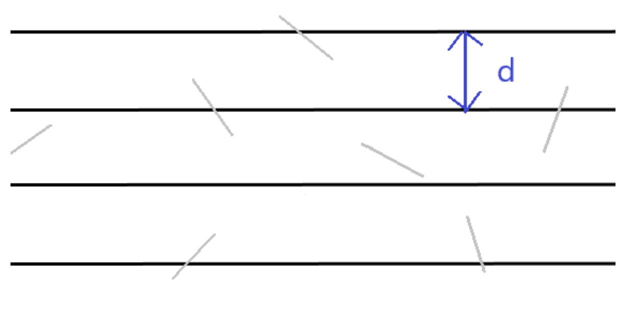

Figure 1.

Here, we present two different solutions to this problem, which show how the number π emerges in this question in two different ways. The first is a clever argument by E. Barbier in 1860 which exploits the shortness of the needle to reduce the problem to a much easier one about throwing circles. The second is the standard proof using calculus which, while not as imaginative, can be generalized to solve the problem with long needles. We follow Peter Phelan's presentation of the results which is based on the account in [1].

The other remarkable thing about Lazzarini's experiment was its uncanny level of accuracy - it was so accurate, in fact, that most mathematicians now agree that it was a hoax. While his experiment was perfectly theoretically valid, it turns out to be a very bad way of estimating π - as will be discussed at the end of this note.

2. First approach (E. Barbier)

Let P_i denote the probability that i crossings will occur. By definition, we know that the expected number of crossings E is given by

$$E = P_1 + 2P_2 + 3P_3 + \cdots .$$

(The probability that a needle comes to rest on a line or with one endpoint exactly on a line is exactly 0 so we can ignore these cases).

Now, if the needle is "short" ($d \geq l$), we have $P_i = 0$ for $i > 1$ and hence $E = P_1$, so instead of looking for P_1 directly we can compute expected number of crossings. This is a useful trick because the expectation has nice properties that make it easier to deal with - in particular, we will show that it depends linearly on the length of the needle, and does not change if the needle is bent.

The expected value of a random variable X is denoted $E[X]$. For the purposes of this proof, we denote by $\mathrm{E}(l)$ the expected number of crossings if a needle of length l (not necessarily greater than d) is thrown. We can imagine dividing the needle into a "front part" of length x and a "rear part" of length y (see figure 2).

Figure 2.

Let X and Y be random variables representing the number of crossings if needles of length x and y (respectively) are thrown. Obviously, the total number of crossings is the sum of the number of crossings that occur in the front and rear parts, so by the linearity of expected value we have

$$\mathrm{E}(l) = \mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y] = \mathrm{E}(x) + \mathrm{E}(y).$$

and, by induction, for any $l_1, \cdots, l_n \in \mathbb{R}$

$$\mathrm{E}(l_1 + l_2 + \cdots + l_n) = \mathrm{E}(l_1) + \mathrm{E}(l_2) + \cdots + \mathrm{E}(l_n).$$

This tells us that we can break the needle into two pieces (and possibly stick them back together at funny angles) without changing the expected number of crossings. It also gives monotonicity: if $x > y$, we have $\mathrm{E}(y) = \mathrm{E}(x) + \mathrm{E}(y - x) > \mathrm{E}(x)$ since $E(y - x) > 0$.

With some more work, it also gives linearity. Firstly, for any rational number $\frac{m}{n}$ we have

$$n\,\mathrm{E}\left(\frac{m}{n}\right) = \mathrm{E}\left(n.\frac{m}{n}\right) = E(m) = mE(1)$$

and hence $E\left(\frac{m}{n}\right) = \frac{m}{n}E(1)$. Then, for any real number r, we use the fact that there exists increasing and decreasing sequences of rational numbers, denoted $(s_n)_{n\geq 0}$ and $(t_n)_{n\geq 0}$ respectively, such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = r$. By monotonicity, for every n we have

$$\mathrm{E}(s_n) \leq \mathrm{E}(r) \leq \mathrm{E}(t_n)$$

or

$$s_n\,\mathrm{E}(1) \leq \mathrm{E}(r) \leq t_n E(1)$$

so

$$\lim_{n\to\infty} s_n\,\mathrm{E}(1) \leq \mathrm{E}(r) \leq \lim_{n\to\infty} t_n\,\mathrm{E}(1)$$

and hence

$$E(r) = \lim_{n\to\infty} s_n\,\mathrm{E}(1) =\leq \lim_{n\to\infty} t_n\,\mathrm{E}(1) = rE(1).$$

Now comes the clever part. Imagine drawing a circle C of radius d on the paper - such a circle must touch a line exactly twice. Choose some n and inscribe inside this circle a regular n-gon Q_n with vertices touching the circle, and outside the circle construct a second regular n-gon Q^n with edges tangent to the circle. Let

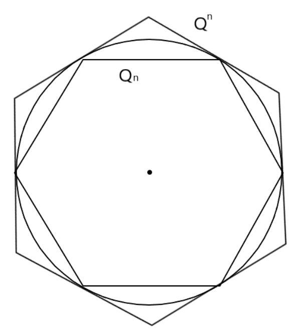

FIGURE 3.

$l(Q_n)$, $l(Q^n)$ denote the lengths of the perimeters of these shapes. Observe that if C touches a line than Q_n touches the same line the same number of times, and likewise with Q_n and C. Therefore Q_n touches a line at most twice, and Q^n touches a line at least twice, so for every n we have

$$\mathrm{E}(l(Q_n)) \leq 2 \leq \mathrm{E}(l(Q^n))$$

or, by linearity,

$$l(Q_n)\,\mathrm{E}(1) \leq 2 \leq l(Q^n)\,\mathrm{E}(1).$$

However, we know that

$$\lim_{n\to\infty} l(Q_n) = \lim_{n\to\infty} l(Q^n) = \pi d,$$

so we conclude $\mathrm{E}(1) = \frac{2}{\pi d}$ and hence that $E(l) = \frac{2l}{\pi d}$.

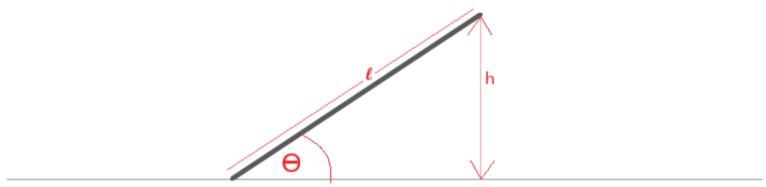

FIGURE 4.

3. Second approach

The more powerful (but perhaps less interesting) way of approaching this problem is to use calculus. Consider the "height" h of the needle when it lands - that is, the vertical space covered by the needle. This depends on θ, by which we denote the smallest angle that the needle makes with the horizontal (see figure 4).

Obviously $0 \leq \theta \leq \frac{\pi}{2}$. A needle which lands at an angle θ to the horizontal will have height $l \sin\theta$, and therefore, if the needle is short, the probability that it intersects one of the lines is $\frac{l \sin\theta}{d}$. (To see this, consider the distance between the highest point of the needle and the line directly below it: the needle will cross if and only if the tip of the needle is within distance $h = l\sin\theta$ of the line).

Therefore (since we assume θ is totally random) the probability we are looking for is given by

$$P = \frac{1}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{l \sin\theta}{d}\, d\theta = \frac{2l}{\pi d} \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = \frac{2l}{\pi d}$$

This method also works for long needles: as above, the probability of a crossing is $\frac{l\sin\theta}{d}$ if $l \sin\theta < d$ and is equal 1 otherwise, so the probability is

$$\begin{aligned} P &= \frac{2}{\pi}\left(\int_0^{\arcsin d/l} \frac{l \sin\theta}{d}\, d\theta + \int_{\arcsin d/l}^{\frac{\pi}{2}} 1 \, d\theta\right) \\ &= 1 + \frac{2}{\pi}\left(\frac{l}{d}\left(1 - \sqrt{1 - \frac{d^2}{l^2}} - \arcsin\frac{d}{l}\right)\right). \end{aligned}$$

4. How to (theoretically) estimate π

This interesting result does indeed give a (theoretically) very simple way of estimating π. One simply needs to find a large piece of paper with parallel lines of distance d apart and a needle (or very thin stick) of length $l \leq d$, and drop it on the paper many times - as did Lazzarini in 1901. The law of large numbers tells us that, if the needle is dropped N times for sufficiently large N, we would expect to see close to $\frac{2lN}{\pi d}$ crossings. Therefore, if M is the number of times we observe a crossing, we can estimate $\pi = \frac{2ln}{dM}$. Using this method, out of 3048 trials with a needle and paper such that $\frac{l}{d} = \frac{5}{6}$, Lazzarini observed 1808 crossings, yielding an extraordinarily accurate estimate of $\pi \approx 3.1415929$!

It turns out, however, that the theory does not in this case translate so nicely to the real world. It has been estimated (see [2]) that, in order to reliably obtain n digits of accuracy, one requires around 10^{2n+2} trials - so in order to get five decimal places (let alone Lazzarini's six), one should expect to have to drop approximately one million million needles (also one would also need to be in possession of extraordinarily fine needles). Could Lazzarini have been lucky? This seems unlikely - if he had observed even one fewer crossing, his estimate would have differed *in the third decimal place* - and even this level of accuracy would have been highly improbable. Also, there is a well-known approximation of $\pi \approx \frac{355}{113}$ which makes Lazzarini's strange choice of 3048 trials look rather suspicious - one should note that $\frac{5}{6}(3048)$ is a multiple of 355. Indeed, considering the other results in Lazzarini's paper, it has been estimated (again, see [2]) that the probability of all of his results and guesses being as accurate as they were - if obtained fairly - to be somewhere in the region of one in ten million.

Unfortunately, then, we must conclude that Lazzarini was either very lucky or (perhaps more likely) very dishonest. The experiment as described - while theoretically quite pleasing - seems to me to be a singularly poor way of approximating π to any reasonable degree of accuracy (even though it is recommended on the wikihow page "How to calculate pi") - I can imagine few things more torturous than dropping a thin needle on a piece of paper several billion times (probabilistically speaking, you are all but guaranteed to stab yourself at least once). Even with a computer, there are far more effective ways of computing π - for those of you who are interested, I will personally recommend using Ramanujan's delightful formula

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}}.$$

That should get you to within six decimal places in three iterations! (Assuming your computer can handle the numbers). A quick Wikipedia search will reveal even better methods that don't involve computing ridiculously high powers of 2. If you are interested in finding out more about Lazzarini's hoax - and how you, too, can fix the experiment to get unfairly accurate estimations of π, I will also recommend consulting [2].

References

[1] Martin Aigner and Günter M. Ziegler. *Proofs from The Book.* Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.

[2] T. H. O'Beirne. *Puzzles and Paradoxes.* Oxford University Press, London, 1965.