THE PIGEON-HOLE PRINCIPLE AND DOUBLE COUNTING

VINCENT GÉLINAS

1. Introduction

The pigeon-hole principle and the principle of double counting are elementary results from combinatorics, whose statements and proofs can be thought of as obvious and yet can be applied to derive fairly non-obvious results. The purpose of this note is as an introduction to these two principles and their applications, beginning with simple examples and eventually reaching more sophisticated results. We end the note with an elementary combinatorial proof of the Brouwer fixed point theorem due to Sperner. The material below consists mostly of highlights from [1, Chapter 27], with main difference being our presentation of the proof of Sperner's Lemma which follows the argument presented in Pedro Tamaroff's talk.

2. The pigeon-hole principle

Principle. If n pigeons are placed into r pigeon-holes and $n > r$, then one pigeon-hole contains more than one pigeon.

This pigeon-hole principle has a myriad of immediate applications. Moreover, one often appeals to a more quantitative version which we record below. Recall that we write $\lceil x \rceil$ for the smallest integer greater than $x \in \mathbb{R}_{\geq 0}$.

Proposition 2.1. *If n pigeons are placed into r pigeon-holes and $n > r$, then one pigeon-hole contains $\geq \lceil \frac{n}{r} \rceil$ pigeons.*

Proof. If each pigeon-hole contains $< \lceil \frac{n}{r} \rceil$ pigeons, then in particular it contains $< \frac{n}{r}$ pigeons. We then have $n < r(\frac{n}{r}) = n$ pigeons in total, a contradiction. □

We will give some applications of the pigeon-hole principle, starting with simple statements, and gradually increasing the sophistication. The next two propositions are fairly simple applications of the principle.

Proposition 2.2. *Let $S \subseteq \{1, 2, \ldots, 2n\}$ be a set of $n+1$ natural numbers. Then one can find two natural numbers in S which are coprime.*

Proof. It suffices to show that S contains a subset $\{k, k+1\}$ consisting of two consecutive numbers. The set

$$\{1, 2, \ldots, 2n\} = \{1, 2\} \cup \{3, 4\} \cup \cdots \cup \{2n-1, 2n\}$$

is a union of n such subsets, and S contains $n+1$ elements. Distributing the elements of S, we are done by the pigeon-hole principle. □

There is a nice variation on the previous example.

Proposition 2.3. *Let $S \subseteq \{1, 2, \ldots, 2n\}$ be a set of $n+1$ natural numbers. Then one can find two natural numbers in S such that one divides the other.*

Proof. Given a natural number $k \in S$, we may write it in the form $k = 2^e m$ for m odd and $e \in \mathbb{Z}_{\geq 0}$, and refer to m as the odd part of k.

The odd number m must lie in the set $\{1, 3, \ldots, 2n-1\}$ of cardinality n. Sending an element $k \in S$ to its odd part, we obtain two elements of S with the same odd part since $|S| = n + 1 > n$, and so one must divide the other. □

Moving on to more involved examples, we first mention an application of the pigeon-hole principle to the study of sums of integer sequences. The following is attributed to Andrew Vázsonyi and Marta Sved.

Proposition 2.4 (Vázsonyi-Sved). *Let $a_1, \ldots, a_n$ be integers, possibly repeated. Then there is a subset of consecutively indexed integers $a_{k+1}, a_{k+2}, \ldots, a_l$ such that the sum*

$$a_{k+1} + a_{k+2} + \cdots + a_l$$

is a multiple of n.

Proof. Let $S = \{0, 1, \ldots, n\}$ and $R = \{0, 1, \ldots, n-1\}$ and consider the map $f : S \to R$ determined by the remainder of $f(m) = a_1 + a_2 + \cdots + a_m$ modulo n. By the pigeon-hole principle, we have $f(k) = f(l)$ for some pair $k < l$ in S, and so

$$\sum_{i=k+1}^{l} a_i = \sum_{i=1}^{l} a_i - \sum_{i=1}^{k} a_i$$

reduces to zero modulo n. The set $a_{k+1}, a_{k+2}, \ldots, a_l$ then has the wanted property. □

We end this section with an interesting result coming from Ramsey problems. Ramsey theory is concerned in general with finding ordered substructures inside sufficiently large unordered sets.

Proposition 2.5 (Erdős-Szekeres). *Let $n, m \in \mathbb{N}$ and consider a sequence of $nm+1$ distinct real numbers $\underline{x} = (x_1, \ldots, x_{nm+1})$. Then in the sequence $\underline{x}$, either there is an increasing subsequence of length $m+1$*

$$x_{i_1} < x_{i_2} < \cdots < x_{i_{m+1}} \qquad (i_1 < i_2 < \cdots < i_{m+1}),$$

a decreasing subsequence of length $n+1$

$$x_{j_1} > x_{j_2} > \cdots > x_{j_{m+1}} \qquad (j_1 < j_2 < \cdots < j_{m+1}),$$

or both.

Proof. Let $S = \{x_1, \ldots, x_{nm+1}\}$ be our set of $nm+1$ real numbers, and define the function $L : S \to \mathbb{N}$ such that $L(x_i)$ is the length of the longest increasing subsequence of $\underline{x}$ starting at x_i. If for some x_i we have $L(x_i) \geq m+1$ then our claim holds, and so assume to the contrary that L takes value in $\{1, 2, \ldots m\}$.

By the pigeon-hole principle, we know that there exists an $s \in \{1, 2, \ldots, m\}$ such that $|L^{-1}(s)| \geq \lceil \frac{mn+1}{m} \rceil = n+1$. Taking $n+1$ distinct elements $x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}} \in L^{-1}(s)$, indexed such that $j_1 < j_2 < \cdots < j_{n+1}$, we claim that the subsequence $(x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}})$ of $\underline{x}$ is then decreasing, which will establish our claim.

If not, then $x_{j_k} < x_{j_{k+1}}$ for some $1 \leq k \leq n$. Since $L(x_{j_{k+1}}) = s$, there is an increasing subsequence of $\underline{x}$ of maximal length s starting at $x_{j_{k+1}}$, and therefore an increasing subsequence starting at x_{j_k} of length $s+1$. This contradicts $L(x_{j_k}) = s$ and the definition of L. □

3. Double counting, or counting in two different ways

Consider finite sets A, B, and a subset $S \subseteq A \times B$ of its Cartesian product. Whenever $(a,b) \in S$, we say that $a \in A$ and $b \in B$ are incident. Given $b \in B$, let us denote the number of elements of A incident to b as $|A|_b$, and similarly define the number $|B|_a$ for $a \in A$. Formally:

$$\begin{aligned} |A|_b &= |\{a \in A \mid (a,b) \in S\}| \\ |B|_a &= |\{b \in B \mid (a,b) \in S\}|. \end{aligned}$$

The principle of double counting is encapsulated in the following equality:

$$\sum_{a \in A} |B|_a = |S| = \sum_{b \in B} |A|_b.$$

One can picture this equality in the following way. For $(a,b) \in A \times B$, define the coefficient

$$m_{a,b} = \begin{cases} 1 & (a,b) \in S \\ 0 & (a,b) \notin S. \end{cases}$$

This gives a matrix $M = (m_{a,b})_{a \in A, b \in B}$ indexed by the elements of $A \times B$. Then $|B|_a$ is the sum of the a-th row, $|A|_b$ is the sum of the b-th column, and the sum of all entries in the matrix is $|S|$. The above equality corresponds to the two ways of adding the entries of M, either by adding the sums of rows first, or by adding the sums of columns.

We next give a simple application of double counting. Recall that a graph $G = (V, E)$ consists of a set of vertices V, and a set of edges $E \subseteq V^{\{0,1\}}$, consisting of two-elements subsets of V.[1] We think of E as lines joining two distinct vertices in V. We say that G is finite if V is a finite set. Given a vertex $v \in V$, its degree is the number of edges containing v, denoted $\deg(v)$.

Proposition 3.1 (Degree sum formula)**.** *Let $G = (V, E)$ be a finite graph. Then we have*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

In particular, the number of vertices of odd degree is even.

Proof. Let $S \subseteq V \times E$ be the set of pairs (v, e) with v a vertex of e. Summing up vertices first, each vertex occurs as many times as its degree and so

$$\sum_{v \in V} \deg(v) = |S|.$$

Summing up edge first, note that each edge e has exactly two end vertices, and so

$$2|E| = |S|.$$

Equating them, we obtain the stated formula. To see the final claim, reduce the degree sum formula modulo 2 and note that odd degree vertices contribute 1, while even degree vertices do not contribute. □

[1] More precisely, what we have defined is an undirected, simple graph G.

4. The Brouwer Fixed Point Theorem and Sperner's Lemma

The celebrated Brouwer fixed point theorem is stated as follows.

Theorem 4.1 (Brouwer, 1912)**.** *Let $f : B^n \to B^n$ be a continuous map from the n-dimensional Euclidean unit ball to itself. Then f has a fixed point, that is $f(x) = x$ for some $x \in B^n$.*

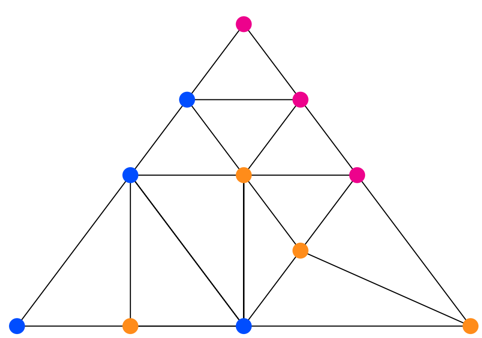
In dimension one this follows from the intermediate value theorem, but modern proofs in higher dimension typically proceed using machinery not available in 1912, such as simplicial homology. It is quite impressive that one can give an elementary combinatorial proof of this theorem, as was done by Sperner in 1928 at the age of 23, using what is now known as Sperner's Lemma. In this section, working in dimension $n = 2$ only, we will give a proof of this lemma and show how to obtain Brouwer's fixed point theorem as a corollary.

We begin with some preliminaries before stating the lemma. Fix a triangle Δ with vertices (v_1, v_2, v_3), and note that in dimension 2 the ball B^2 is homeomorphic Δ. It then suffices to prove that any continuous map $f : \Delta \to \Delta$ has a fixed point. We will analyse the behavior of f by subdividing Δ into smaller and smaller triangles.

A **triangulation** of Δ is a finite decomposition of Δ into smaller triangles, which fit together edge-by-edge. Fixing a triangulation of Δ, assume that we have colored the vertices of the smallest triangles from a set of "colors" $\{1, 2, 3\}$, done according to the following rules:

- The vertex v_i receives the color i.
- Any vertex on the edge between v_i, v_j has color in $\{i, j\}$.
- The interior vertices are colored arbitrarily.

Given a triangulation of Δ, we will refer to a coloring respecting the above rules as a Sperner coloring. The example below is a Sperner coloring, where we use (blue, orange, magenta) for the colors $(1, 2, 3)$:



Lemma 4.2 (Sperner's Lemma)**.** *Let Δ have a triangulation whose vertices have a Sperner coloring. Then one of the triangles in the triangulation is 3-colored, that is, has all vertices of different colors.*

Proof. First, we say that an edge is $\{1,2\}$-colored if its vertices have one of each color in $\{1,2\}$. We will traverse all edges which are $\{1,2\}$-colored, in the hope of arriving at a 3-colored triangle. Note that the outside edges of our triangulation between v_1 and v_2 have vertices colored by the set $\{1,2\}$ by our coloring rule. Let E be the set of $\{1,2\}$-colored edges between v_1, v_2, whose cardinality is odd by a simple induction.

Entering an edge in E and traversing $\{1,2\}$-colored edges without backtracking, one of three things can happen:

1) We visit a triangle more than once.
2) We eventually emerge at a different edge in E.
3) We eventually reach a dead-end.

We claim that 1) cannot happen. If it does, then there is a first revisited triangle in our path. We previously entered this triangle via a first $\{1,2\}$-colored edge, left via a second $\{1,2\}$-colored edge, and must now be entering via the third edge by our assumption. But this also cannot be $\{1,2\}$-colored.

Therefore we never revisit triangles, and either exit through another edge in E or reach a dead-end, which must be a 3-colored triangle. We finish by noting that the in-and-out paths pair up an even numbers of edges in E, which had odd cardinality, and so an odd number of edges must lead to a dead-end. □

Remark 4.3. It seems that we have not used any of the ideas developed in the previous sections. Is this really the case?

We now turn to the proof of Brouwer's Theorem in dimension 2. We will denote by $\mathcal{T}$ a given triangulation of Δ, and let $\delta(\mathcal{T})$ stand for the maximal length of edges of small triangles in the triangulation $\mathcal{T}$. Note that we can always construct a sequence of triangulations $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k, \dots$, each refining the previous one, such that $\lim_{k\to\infty} \delta(\mathcal{T}_k) = 0$. For instance, one add a vertex to any small triangle at its centre of mass and join it to its vertices.

Corollary 4.4 (Brouwer)**.** *Any continuous map $f : \Delta \to \Delta$ has a fixed point.*

Proof. It's enough to do this for any specific triangle, and we pick $\Delta \subseteq \mathbb{R}^3$ with vertices $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ for bookkeeping purposes.

Pick a sequence of triangulations $\mathcal{T}_k$ with $\lim_{k\to\infty}\delta(\mathcal{T}_k) = 0$. For the triangulation $\mathcal{T}_k$, we color its vertices $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ by the color $\lambda(v) := \min\{i \mid f(v)_i < v_i\}$. This color is defined so long as $f(v) - v$ has some negative coordinate. If $f(v)_i \geq v_i$ for $i = 1,2,3$, we claim that v must be a fixed point. To see this, note that $\Delta \subseteq \mathbb{R}^3$ lies in the hyperplane $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$ and so $\sum_{i=1}^{3} (f(v)_i - v_i) = 0$. We deduce that $f(v) - v$ in fact has both a negative and positive coordinate, unless v is a fixed point.

If no vertex of $\mathcal{T}_k$ is a fixed point, we claim that this coloring satisfies the hypothesis of Sperner's lemma. The only negative coordinate of $f(e_i) - e_i$ is the i-th, and so $\lambda(e_i) = i$. Next, if the vertex $v \in \Delta$ lies on the edge opposite e_i, then its i-th coordinate is zero and so $f(v) - e_i$ has non-negative i-th coordinate, which gives $\lambda(v) \neq i$. This verifies the hypothesis.

Sperner's Lemma guarantees a 3-colored triangle in $\mathcal{T}_k$, with vertices $v^{k:1}, v^{k:2}, v^{k:3}$ colored as $\lambda(v^{k:i}) = i$, say. The sequence of points $(v^{k:1})_{k\geq 1}$ in the compact set Δ has a convergent subsequence, and replacing $\{\mathcal{T}_k\}_{k\geq 1}$ with the corresponding subsequence we can assume that $(v^{k:1})_{k\geq 1}$ converges to some point $v \in \Delta$. Since the edge distance $\delta(\mathcal{T}_k) \to 0$ goes to zero, the sequences $(v^{k:2})_{k\geq 1}$, $(v^{k:3})_{k\geq 1}$ also converge to v. We claim that $f(v) = v$.

Since $\lambda(v^{k:1}) = 1$, $f(v)_1 - v_1^{k:1} < 0$, and similarly $f(v)_i - v_i^{k:i} < 0$ for $i = 2, 3$. By continuity we have $f(v)_i - v_i \leq 0$ for all $i = 1, 2, 3$. But this contradicts that $f(v) - v$ must have both a negative and positive coordinate, unless v is a fixed point. $\square$

References

[1] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.