Assignment 7

MA341C — Seminar on *Proofs from THE BOOK*
Trinity College Dublin

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note: solutions to this assignment are due by 11am on Wednesday, November 7th. Please attach a cover sheet with a declaration (http://tcd-ie.libguides.com/plagiarism/declaration) confirming that you know and understand College rules on plagiarism. All exercises are weighed equally unless otherwise stated.

A *chain* in a partially ordered set is a subset of pairwise comparable elements. An *antichain* in a partially ordered set is a subset of pairwise incomparable elements. (These notions generalise the notions of chains and antichains presented in class, where the partially ordered set would be the set of all subsets of $\{1,\dots,n\}$ ordered by inclusion.) The celebrated *Dilworth's theorem* states that the maximal size of an antichain in any partially ordered set P is equal to the minimal number of disjoint chains into which one can partition P.

Exercise 1. Show that the family of all $n/2$-sets (for even n) or, respectively, the two families of all $(n-1)/2$-sets and $(n+1)/2$-sets (for odd n) are the only antichains of the maximal possible size. (*Hint*: show that when n is odd, if $X \subset Y$ are, respectively, an $(n-1)/2$-set and an $(n+1)/2$-set, then any antichain of maximal size must contain either X or Y, and use this to obtain a contradiction.)

Exercise 2. Use Dilworth's theorem to prove the following results:

1. If each chain in a partially ordered set P is of size at most m and each antichain is of size at most n, then the number of elements in P is at most mn.
2. Every sequence of $nm+1$ distinct real numbers has either an increasing subsequence of size n, or a decreasing subsequence of size m. (*Hint*: use this sequence to define a partially ordered set P in which decreasing subsequences become antichains.)

Exercise 3. Prove the dual of Dilworth's theorem: the maximal size of a chain in a finite partially ordered set P is equal to the minimal number of disjoint antichains into which one can partition P. (*Hint*: suppose that for two elements $x \neq y \in P$, the longest chain with the maximal element x is as long as the longest chain with the maximal element y. Show that x and y are incomparable.)

Exercise 4. Show that for a given d, the quantity

$$p = 1 + \frac{2}{\pi}\left(\frac{\ell}{d}\left(1 - \sqrt{1 - \frac{d^2}{\ell^2}}\right) - \arcsin\frac{d}{\ell}\right),$$

viewed as a function of $\ell \geq d$, is an increasing function that tends to 1 as $\ell \to \infty$.