Assignment 5

MA341C — Seminar on *Proofs from THE BOOK*

Trinity College Dublin

NAME AND SURNAME: ....

STUDENT NUMBER: .... NUMBER OF PAGES: ....

Note: solutions to this assignment are due by 11am on Wednesday, October 17th. Please attach a cover sheet with a declaration (http://tcd-ie.libguides.com/plagiarism/declaration) confirming that you know and understand College rules on plagiarism. All exercises are weighed equally unless otherwise stated.

Exercise 1. Let n be an integer. Show that there exists a positive integer A (depending on n) such that for every integer c, prime divisors of $\phi_n(c)$ are either smaller than A or congruent to 1 modulo n. (*Hint:* the cyclotomic polynomial $\phi_n(x)$ is coprime to the product of the polynomials $x-1$, x^2-1, ..., $x^{n-1}-1$ in $\mathbb{Q}[x]$, so the ideal generated by them coincides with $\mathbb{Q}[x]$. Deduce that the ideal generated by those polynomials in $\mathbb{Z}[x]$ contains an integer A, and take it from there.)

Exercise 2. Suppose that A is a set of subsets of $\mathbb{N}$ such that whenever $S_1, S_2 \in \mathsf{A}$, we have $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. Is it true that the set A must be countable? (*Hint:* it is better to identify $\mathbb{N}$ with $\mathbb{Q}$.)

Exercise 3. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that the monoid $\langle A, B \rangle$ generated by A and B under matrix multiplication is the free monoid on two generators. (*Hint:* Relate this monoid to the Calkin-Wilf tree.)

Exercise 4. The unique polynomial $T_n(x)$ of degree n such that $T_n(\cos(\vartheta)) = \cos(n\vartheta)$ is the n-th Chebyshev polynomial (of the first kind). The first few Chebyshev polynomials are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, ...

a) Prove the "multiplicative formula" $T_m(T_n(x)) = T_{mn}(x)$.

b) Show that the polynomials $T_n(x)$ satisfy the recurrence relation $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$, and deduce that for all $n \geq 1$, we have

$$T_n(x) = \det \begin{pmatrix} 2x & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2x & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2x & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2x & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & x \end{pmatrix}.$$