MA 3416: Group representations
Homework problems due April 2, 2015

1. Consider S_4 as a subgroup in S_5 and define a function ψ on S_5 by the formula

$$\psi(g) = \frac{1}{24} \sum_{\substack{h \in S_5, \\ hgh^{-1} \in S_4}} \chi_U(hgh^{-1}),$$

where U denotes the 2-dimensional complex irreducible representation of S_4; if the sum is over an empty set, it is considered to be equal to zero. Prove that ψ is a character of some complex representation of S_5, and find multiplicities of irreducibles in that representation.

2. (**a**) Does there exist a finite group which has precisely four one-dimensional complex representations, precisely one five-dimensional complex irreducible representation, and no other complex irreducible representations?

(**b**) Using the fact that that the number of elements in the conjugacy class of $g \in G$ is equal to $\frac{\#G}{\#C_g^G}$, where C_g^G is the *centraliser* of g, prove that all finite groups with three conjugacy classes are $\mathbb{Z}/3\mathbb{Z}$ and S_3.

(**c**) Prove that though $1 + 5^2 + 13^2$ is divisible by 5 and 13, there is no finite group which has just three complex irreducible representations whose dimensions are 1, 5, and 13.

3. Let us consider the set representation U_k of S_n that arises from the action of S_n on the set of all k-element subsets of $\{1, 2, \ldots, n\}$.

(**a**) Prove that $U_k \simeq U_{n-k}$.

(**b**) Compute the dimension of the space of intertwining operators $\mathrm{Hom}_{S_n}(U_k, U_l)$ for all k and l.

4. In the notation of the previous question, show that the ring of intertwining operators $\mathrm{Hom}_{S_n}(U_k, U_k)$ (where addition and multiplication is addition and composition of linear transformations respectively) is commutative.