MA 3416: Group representations
Homework problems due February 2, 2015

1. Prove that over a field of characteristic different from 2 the symmetric group S_n has precisely two one-dimensional representation: the trivial representation and the sign representation. (*Hint*: it was already done in class for $n = 3$; generalise that approach).

2. Describe all one-dimensional complex representations of the group (**a**) D_4; (**b**) Q_8.

3. The group A_4 of all even permutations of 4 elements has order $12 = 24/2$. Describe all one-dimensional representations of that group.

4. Find in the dihedral group D_n (group of symmetries of the regular n-gon, consisting of the unit elements, $n-1$ nontrivial rotations, and n mirror reflections) two elements a, b that generate this group and satisfy relations $a^n = e$, $b^2 = e$, and $ba = a^{-1}b$. Use these elements to describe all 1-dimensional complex representations of D_n.

5. Show that any irreducible complex representation of a finite abelian group is one-dimensional.

6. Show that setting $\rho_\pm(\bar{1})$ to be the counterclockwise rotation of $\mathbb{R}^2$ about the origin through $\pm 120°$ defines two real 2-dimensional representations of the cyclic group $\mathbb{Z}/3\mathbb{Z}$. Show that these representations $(\mathbb{R}^2, \rho_+)$ and $(\mathbb{R}^2, \rho_-)$ are irreducible over real numbers.

7. Describe all homomorphisms between the representations $(\mathbb{R}^2, \rho_+)$ and $(\mathbb{R}^2, \rho_-)$ from the previous question, and determine whether these representations are isomorphic.

8. Consider $G = S_n$, $V = \mathbb{C}^n$, and define a complex representation of G as follows: $\rho(\sigma)e_i = e_{\sigma(i)}$. Show that V is equivalent to the direct sum of the trivial representation and an irreducible representation of dimension $n-1$. (*Hint*: show that the space of intertwining operators $\varphi\colon V \to V$ is two-dimensional, and use that fact to establish this statement).