UNIVERSITY OF DUBLIN

XMA3413

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JS Mathematics **2012/13**
SS Mathematics

MODULE 3413: SAMPLE EXAM

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For any task, the number of points you can get for a complete solution of this task is printed next to it.
For your convenience, character tables for A_4, A_5, S_4, and S_5 are included; see the last page.
Unless otherwise stated, all groups are finite, and all representations are complex and finite dimensional

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. (20 points) Define a *group representation*. What is meant by saying that two representations (V, ρ) and (W, π) are isomorphic?

   Define the direct sum of two representations (V, ρ) and (W, π) of the same group G.

   Prove that any finite dimensional complex representation of a finite group G is isomorphic to a direct sum of irreducible representations of that group.

2. (20 points) What is meant by saying that a representation (V, ρ) of a group G is a *set representation*?

   Consider the set representation U_k of S_n that corresponds to the action of S_n on the set of all k-element subsets of $\{1, 2, \ldots, n\}$.

   Prove that $U_k \simeq U_{n-k}$.

   Compute the dimension of the space of intertwining operators $\mathrm{Hom}_{S_n}(U_k, U_l)$.

3. (20 points) Consider S_4 as a subgroup in S_5 and define a function ψ on S_5 by the formula

$$\psi(g) = \frac{1}{24} \sum_{\substack{h \in S_5, \\ hgh^{-1} \in S_4}} \chi_U(hgh^{-1}),$$

   where U denotes the 2-dimensional irreducible representation of S_4 (see the character table); if the sum is over an empty set, it is considered to be equal to zero. Prove that ψ is a character of some representation of S_5, and find multiplicities of irreducibles in that representation.

4. (20 points) Recall that the nth exterior (wedge) power of a vector space W (which is denoted by $\Lambda^n(W)$) is a subspace in its nth tensor power $W^{\otimes n}$ which is spanned by all skew-symmetric products

$$w_1 \wedge w_2 \wedge \ldots \wedge w_n = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \ldots \otimes w_{\sigma(n)}$$

   for all $w_1, \ldots, w_n \in W$. The nth exterior power $\Lambda^n(A)$ of an operator $A \colon W \to W$ is defined by

$$\Lambda^n(A)(w_1 \wedge w_2 \wedge \ldots \wedge w_n) = (Aw_1) \wedge (Aw_2) \wedge \ldots \wedge (Aw_n).$$

If (W, ρ) is a representation of a finite group G, $(\Lambda^n(W), \Lambda^n(\rho))$ is its subrepresentation which is called the nth exterior power of the representation W.

For a representation V of a group G, prove that characters of V and $\Lambda^2(V)$ are related by

$$\chi_{\Lambda^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2)).$$

Find multiplicities of irreducibles in $\Lambda^2(U_1)$ and in $\Lambda^2(U_2)$, where U_i are the 3-dimensional irreducible representations of A_5.

5. (20 points) Does there exist a finite group which has precisely four one-dimensional representations, precisely one five-dimensional irreducible representation, and no other irreducible representations?

   Using the fact that that the number of elements in the conjugacy class of $g \in G$ is equal to $\frac{\#G}{\#C_g}$, where C_g is the *centraliser* of g (the subgroup of all $h \in G$ such that $gh = hg$), prove that all finite groups with three conjugacy classes are $\mathbb{Z}/3\mathbb{Z}$ and S_3.

   Prove that though $1 + 5^2 + 13^2$ is divisible by 5 and 13, there is no finite group which has just three irreducible representations whose dimensions are 1, 5, and 13.

Appendix: character tables of A_4, A_5, S_4, and S_5.

Notation: the top row of each table lists conjugacy classes; for S_4 and S_5 conjugacy classes are encoded by lengths of cycles, for A_4 and A_5 we use subscripts for splitting classes; for example 31_1 and 31_2 are two classes consisting of 3-cycles. The second row indicates cardinalities of conjugacy classes. Further rows list irreducble characters (everywhere below $\omega = \frac{-1+\sqrt{-3}}{2}$, $\tau = \frac{1+\sqrt{5}}{2}$). We use the notation $\mathbb{1}$ for the trivial representation, and sgn for the sign representation. For symmetric groups S_n, V denotes the nontrivial summand of the permutation representation in $\mathbb{C}^n$. For any representation M of S_n, we let $M' = M \otimes \mathrm{sgn}$.

	1^4	2^2	31_1	31_2
#	1	3	4	4
$\mathbb{1}$	1	1	1	1
R_1	1	1	ω	ω^2
R_2	1	1	ω^2	ω
V	3	-1	0	0

	1^5	$2^2 1$	31^2	5_1	5_2
#	1	15	20	12	12
$\mathbb{1}$	1	1	1	1	1
U_1	3	-1	0	τ	$-\frac{1}{\tau}$
U_2	3	-1	0	$-\frac{1}{\tau}$	τ
V	4	0	1	-1	-1
W	5	1	-1	0	0

	1^4	21^2	2^2	31	4
#	1	6	3	8	6
$\mathbb{1}$	1	1	1	1	1
sgn	1	-1	1	1	-1
U	2	0	2	-1	0
V	3	1	-1	0	-1
V'	3	-1	-1	0	1

	1^5	21^3	31^2	$2^2 1$	41	32	5
#	1	10	20	15	30	20	24
$\mathbb{1}$	1	1	1	1	1	1	1
sgn	1	-1	1	1	-1	-1	1
V	4	2	1	0	0	-1	-1
V'	4	-2	1	0	0	1	-1
W	5	1	-1	1	-1	1	0
W'	5	-1	-1	1	1	-1	0
$\Lambda^2 V$	6	0	0	-2	0	0	1