MA2317: Introduction to Number Theory
Homework problems due December 16, 2010

Note the unusual deadline: this assignment is due on Thursday, December 16!

The last three questions are optional. Other questions are assessed.

1. Solve the congruence (**a**) $x^2 \equiv -3 \pmod{13}$; (**b**) $x^2 \equiv -3 \pmod{169}$; (**c**) $x^2 \equiv -3 \pmod{2197}$.

2. Compute the p-adic expansions of (**a**) $\frac{1}{10}$ in $\mathbb{Z}_{11}$; (**b**) $-\frac{9}{16}$ in $\mathbb{Z}_{13}$.

3. Compute the p-adic expansions of (**a**) $\frac{1}{24}$ in $\mathbb{Q}_2$; (**b**) $\frac{1}{120}$ in $\mathbb{Q}_5$.

4. Show that for every p the polynomial $(x^3 - 37)(x^2 + 3)$ has roots in $\mathbb{Z}_p$. (*Hint*: if $p \not\equiv 1 \pmod 3$, the mapping $x \mapsto x^3$ of the group of nonzero remainders modulo p to itself is injective, so every element has to be a cube.)

5. Show that for every p the equation $x^2 + 2y^4 - 17z^4 = 0$ has nonzero solutions in $\mathbb{Z}_p$.

6. Show that as n tends to infinity, the rational number

$$a_n = 2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots + \frac{2^n}{n}$$

tends to zero in 2-adic metric. In other words, if we write $a_n = 2^{c_n}\frac{p_n}{q_n}$ with odd p_n and q_n, then c_n tends to infinity as n tends to infinity.

7. Is the following proof of irrationality of π correct and rigorous? Why?

> Assume that $\pi = \frac{a}{b}$, and let $p \neq 2$ be a prime number not dividing a. Then
>
> $$0 = \sin(pb\pi) = \sin(pa) = \sum_{n\geqslant 0} \frac{(-1)^n (pa)^{2n+1}}{(2n+1)!} \equiv pa \pmod{p^2},$$
>
> and we have a contradiction.