MA2317: Introduction to Number Theory
Homework problems due October 29, 2010

The last two questions are optional. Other questions are assessed.

1. Solve the system of congruences

$$\begin{aligned} x &\equiv 11 \pmod{23}, \\ x &\equiv 12 \pmod{25}, \\ x &\equiv 13 \pmod{27}. \end{aligned}$$

2. For which a does the following system of congruences have integer solutions?

$$\begin{aligned} x &\equiv a \pmod{100}, \\ x &\equiv b \pmod{35}. \end{aligned}$$

3. Compute the following Legendre symbols:
(**a**) $\left(\frac{1192}{1291}\right)$; (**b**) $\left(\frac{499}{1291}\right)$; (**c**) $\left(\frac{2357}{3571}\right)$.

4. Which of the following congruences have solutions?
(**a**) $x^2 - 12x + 31 \equiv 0 \pmod{47}$;
(**b**) $x^2 + 3x - 31 \equiv 0 \pmod{101}$;
(**c**) $x^2 - 12x + 31 \equiv 0 \pmod{235}$?

5. (**a**) Show that for an odd prime p we have $\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$.

(**b**) Use the previous result to prove that for every n all prime divisors of $9n^2+3n+1$ are of the form $3k+1$, and adapt the "$p_1p_2\cdots p_n-1$"-argument proving the infinitude of primes to show that there are infinitely many primes of the form $3k+1$.

6. Show that there are infinitely many prime numbers q such that $2q+1$ is not a prime. (*Hint*: Fermat's Little Theorem might be helpful at some point.)

7. Let p be an odd prime number.

(**a**) Show that the function $k \mapsto \frac{1-k}{1+k}$ maps the set $\mathbb{Z}/p\mathbb{Z} \setminus \{-1\}$ to itself and is a 1-to-1 correspondence.

(**b**) Show that

$$\sum_{k=0}^{p-1} \left(\frac{k}{p}\right) = 0.$$

(**c**) Find the number of solutions to the equation $x^2 + y^2 = 1$ in $\mathbb{Z}/p\mathbb{Z}$. (*Hint*: this number is equal to $\sum_{y=0}^{p-1}(1 + \left(\frac{1-y^2}{p}\right))$.)