MA2215: Fields, rings, and modules
Tutorial problems, November 15, 2012

**1.** Since $175 = 5^2 \cdot 7$, the Eisenstein criterion applies with $p = 7$.

**2.** (**a**) Clearly, no $p$ works for this polynomial since the coefficient of $x$ is equal to $1$. However, $f(x+2) = x^2+5x+10$, so $p = 5$ works. If $f(x) = g(x)h(x)$, then $f(x+2) = g(x+2)h(x+2)$, so irreducibility of $f(x)$ is equivalent to irreducibility of $f(x+2)$.

(**b**) We have $x^4 + 4 + 4x^2 - 4x^2 = (x^2 + 2)^2 - 4x^2 = (x^2 + 2x + 2)(x^2 - 2x + 2)$. Or, if you consider this solution too sneaky, note that we are looking for a factorisation

$$x^4 + 4 = (x^2 + ax + b)(x^2 + cx + d),$$

since this polynomial has no real roots hence no factors of degree $1$, and also the product of leading coefficients is equal to $1$, so they are either both $1$ or both $-1$, and multiplying the factors simultaneously by $-1$, we can assume that both leading coefficients are equal to $1$. Moreover, the coefficient of $x^3$ is zero, so $a + c = 0$. Finally, $bd = 4$, so each of $b$ and $d$ can take values $\pm 1$, $\pm 2$, $\pm 4$. In fact, the roots of our polynomial are complex fourth roots of $-4$, and constant terms $b$ and $d$ are products of pairs of those roots, so we cannot get $\pm 1$ and $\pm 4$ as roots. Hence we should just test factorisations $(x^2 + ax + 2)(x^2 - ax + 2)$ and $(x^2 + ax - 2)(x^2 - ax - 2)$, and the first one works.

**3.** Consider this polynomial as a polynomial in $y$ whose coefficients are polynomials in $x$, that is $y^3 + (x^2 - 1) = y^3 + (x - 1)(x + 1)$. Clearly, the Eisenstein criterion for $R = \mathbb{C}[x]$, $p = x - 1$ applies.

**4.** Suppose that $f(x) = g(x)h(x)$, where the degrees of $g$ and $h$ are $k$ and $4 - k$ respectively. Then $f(1/x) = g(1/x)h(1/x)$, and $x^4 f(1/x) = x^k g(1/x) x^{4-k} h(1/x)$. But $x^4 f(1/x) = 42 - 8x + 12x^2 - 6x^3 - x^4$, and the Eisenstein criterion with $p = 2$ applies.