MA2215: Fields, rings, and modules
Tutorial problems, October 18, 2012

1. Since $\mathbb{Z}/2\mathbb{Z}$ is clearly a field (the only nonzero element $1 = \bar{1}$ is invertible), the ring of polynomials $(\mathbb{Z}/2\mathbb{Z})[t]$ is a Euclidean domain. Explain which of the following polynomials are irreducible: $t, t+1$, t^2, t^2+t, t^2+1, t^2+t+1.

In the two following questions, feel free to use the property $d(zw) = d(z)d(w)$ of the norm of Gaussian integers. This property implies that if z is invertible, then $d(z)$ must be invertible, and also that if $z = z_1 z_2$ is *not* irreducible, then $d(z) = d(z_1)d(z_2)$ is a factorisation of $d(z)$, which helps to find candidates for the factors z_1 and z_2.

2. Show that in $\mathbb{Z}[i]$ the only invertible elements are ± 1 and $\pm i$.

3. Which of the Gaussian integers $2 = 2 + 0i$, $3 + i$, $7i$ are irreducible in $\mathbb{Z}[i]$?

4. Find some greatest common divisor of Gaussian integers $a = 11 + 13i$ and $b = 27 + 31i$.

Optional question (if you have some time left): Let $\omega = \frac{1+\sqrt{-3}}{2}$ be one of the complex roots of the equation $z^2 - z + 1 = 0$. In this question we shall consider the set E of all complex numbers of the form $a + b\omega$, where a, b are integers.

Show that $\omega^6 = 1$, and draw ω on the complex plane.

Show that E is a subring of $\mathbb{C}$. Given an element $z = a + b\omega$ in E, draw on the complex plane the set of all multiples of z in E. (*Hint*: check your notes about $\mathbb{Z}[i]$ from class, and think how the point $\omega \cdot z$ is obtained from z.)

Show that E is a Euclidean domain.