MA2215: Fields, rings, and modules
Homework problems due on December 3, 2012

1. Compute the degree over $\mathbb{Q}$ of the splitting field of $x^4 - 4x^2 - 5$.

2. Compute the degree over $\mathbb{R}$ and over $\mathbb{C}$ of the splitting field of $x^4 - 4x^2 - 5$.

3. Compute the degree over $\mathbb{Q}$ of the splitting field of $x^{11} - 5$. (*Hint*: show that it contains subfields of degrees 10 and 11.)

4. Compute the degree over $\mathbb{F}_3$ of the splitting field of $x^8 + 2$. (*Hint*: $2 = -1$ in $\mathbb{F}_3$.)

5. (**a**) Show that for the polynomial $x^p - x \in \mathbb{F}_p[x]$ we have $f(x) = f(x+1)$.

(**b**) Show that if for a polynomial $f(x) \in \mathbb{F}_p[x]$ we have $f(x) = f(x+1)$, then the degree of f is at least p.

(**c**) Show that $g(x) = x^p - x - 1 \in \mathbb{F}_p[x]$ is irreducible. (*Hint*: show that $g(x) = g(x+1)$, and deduce that if $g(x)$ has a nontrivial factorisation into irreducibles, then for each of those irreducibles $h(x)$ we either have $h(x) = h(x+1)$ or all $h(x)$, $h(x+1)$, $\ldots$, $h(x+p-1)$ are different irreducible factors of $g(x)$. The latter would mean that $g(x)$ factorises into linear factors, hence has roots.)