MA2215: Fields, rings, and modules
Homework problems due on October 1, 2012

1. Explain why $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[t]$ are rings. Do they have unit elements?

2. Show that in the ring of integers, $ab = ac$ implies $b = c$ whenever a is not equal to zero. Give an example of a ring where this statement does not hold.

3. Show that the subset of $\mathbb{Z}[t]$ consisting of all polynomials $f(t)$ such that $f(1) = 0$ is a subring of $\mathbb{Z}[t]$. Show the same for the subset of $\mathbb{Z}[t]$ consisting of all polynomials $f(t)$ such that $f(1) = f'(1) = 0$. Do these subrings have unit elements?

4. Show that for any ring R square R-valued matrices of the given size (notation: $\mathrm{Mat}_n(R)$) form a ring with respect to matrix multiplication.

5. Show that $\mathrm{Mat}_n(R)$ has a unit element if and only if R has a unit element. (*Hint:* if A is the unit element of $\mathrm{Mat}_n(R)$, we have $AX = XA = X$ for matrices $X = \begin{pmatrix} r & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 0 \end{pmatrix}$, where $r \in R$.)