MA2215: Fields, rings, and modules
Homework problems due on October 15, 2012

1. Let us, as suggested by the hint, use the map $\varphi\colon \mathrm{Mat}_n(R) \to \mathrm{Mat}_n(R/I)$,

$$\varphi\left((a_{pq})_{p,q=1,\dots,n}\right) = (a_{pq} + I)_{p,q=1,\dots,n}.$$

This map is a homomorphism because, for instance,

$$\varphi(ab)_{pq} = (ab)_{pq} + I = (\sum_i a_{pi}b_{iq}) + I = \sum_i (a_{pi}b_{iq} + I) =$$
$$= \sum_i (a_{pi} + I)(b_{iq} + I) = (\varphi(a)\varphi(b))_{pq}$$

because of the definition of matrix product and the definition of operations in factor rings. The other properties of homomorphisms are checked similarly. Also, it is clear that $\mathrm{Im}(\varphi) = \mathrm{Mat}_n(R/I)$ (every matrix with the matrix element in row p and column q being the coset $r_{pq}+I$ is the image of the matrix with matrix elements r_{pq}), and that $\mathrm{Ker}(\varphi) = \mathrm{Mat}_n(I)$ (if $r_{pq} + I = 0 + I$ for all p, q, we have $r_{pq} \in I$ for all p, q.

2. Let us, as suggested by the hint, use the map $\varphi\colon R[t] \to (R/I)[t]$,

$$\varphi(a_0 + a_1t + \ldots + a_nt^n) = (a_0 + I) + (a_1 + I)t + \ldots + (a_n + I)t^n.$$

This map is a homomorphism because, for instance, for $f(t) = a_0 + a_1t + \ldots + a_nt^n$ and $g(t) = b_0 + b_1t + \ldots + b_mt^m$ the coefficient of t^k in $\varphi(f(t)g(t))$ is equal to

$$(\sum_{i+j=k} a_ib_j) + I = \sum_{i+j=k} (a_ib_j + I) = \sum_{i+j=k} (a_i + I)(b_j + I),$$

which is the coefficient of t^k of $\varphi(f(t))\varphi(g(t))$ because of the definition of the polynomial product and the definition of operations in factor rings. The other properties of homomorphisms are checked similarly. Also, it is clear that $\mathrm{Im}(\varphi) = (R/I)[t]$ (every polynomial with the coefficient of t^k being the coset $r_k + I$ is the image of the polynomial with coefficients r_k), and that $\mathrm{Ker}(\varphi) = I[t]$ (if $r_k + I = 0 + I$ for all k, we have $r_k \in I$ for all k.

3. Let us, as suggested by the hint, use the map $\varphi\colon R/J \to R/I$, $\varphi(r + J) = r + I$. Since $J \subset I$, this map is well defined, and is a homomorphism: if r_1 and r_2 are in the same coset modulo J then they of course are in the same coset modulo I, and $(rs + I) = (r + I)(s + I)$ in R/I, $(rs + J) = (r + J)(s + J)$ in R/J etc. Also, this map is obviously surjective, since all cosets $r + I$ are in the image by inspection of the formula for φ, and its kernel is the ideal of R/J which consists of all cosets $r + J$ for which $r + I = 0 + I$, so $r \in I$. This ideal is precisely I/J by Second Isomorphism Theorem.

4. (**a**) Yes, since for a nonzero polynomial in R its leading coefficient is $\bar{1}$, so for two nonzero polynomials the leading coefficient of the product is nonzero. (**b**) No, $\bar{2} \cdot \bar{2} = \bar{4} = 0$. (**c**) Yes, since 5 is a prime number, so $\mathbb{Z}/5\mathbb{Z}$ is a field, and the argument from (a) applies. (**d**) No, $(t + 1 + (t^2 - 1)\mathbb{Z}[t])(t - 1 + (t^2 - 1)\mathbb{Z}[t]) = (t^2 - 1 + (t^2 - 1)\mathbb{Z}[t]) = 0$. (**e**) No, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$.