MA2215: Fields, rings, and modules
Homework problems due on October 8, 2012

1. (**a**) Each homomorphism takes 0 to 0, and $\varphi(\overline{2}) = \varphi(\overline{1}+\overline{1}) = \varphi(\overline{1}) + \varphi(\overline{1})$, so we should just determine possible values of $\varphi(\overline{1})$. We have the constrain $\varphi(\overline{1}\cdot\overline{1}) = \varphi(\overline{1})\cdot\varphi(\overline{1})$, so $\varphi(\overline{1})(\varphi(\overline{1})-1) = 0$. Since the target is $\mathbb{Z}/3\mathbb{Z}$, we conclude that the possible values are $\overline{0}$ and $\overline{1}$. In fact, both are fine: for $\varphi(\overline{1}) = \varphi(\overline{0}) = 0$ the four required properties reduce to $0+0=0$, $0\cdot 0 = 0$, $-0=0$, $0=0$, and for $\varphi(\overline{1}) = \overline{1}$ and $\varphi(\overline{0}) = \overline{0}$ the four required properties reduce to $a+b=a+b$, $ab=ab$, $-a=-a$, $0=0$.

(**b**) If we start as above, we see that we only need to define $\varphi(\overline{1})$. Since $\overline{1}+\overline{1}+\overline{1}=\overline{3}=\overline{0}$ in $\mathbb{Z}/3\mathbb{Z}$, we want $\varphi(\overline{1})+\varphi(\overline{1})+\varphi(\overline{1}) = \overline{0}$ to hold. But in $\mathbb{Z}/2\mathbb{Z}$ we have

$$\varphi(\overline{1})+\varphi(\overline{1})+\varphi(\overline{1}) = 3\varphi(\overline{1}) = \varphi(\overline{1}),$$

and we conclude that $\varphi(\overline{1}) = 0$. Therefore, in this case the only map which is a homomorphism sends all elements to zero.

2. Let us, as suggested, consider the map between these rings that takes the coset of $f(x) + (x^2-1)\mathbb{R}[x]$ to the pair of numbers $(f(1), f(-1))$. Let us check that it is well defined. Indeed, if $f(x)$ and $g(x)$ are in the same coset, that is $f(x) = g(x) + h(x)(x^2-1)$, we have $f(1) = g(1)$ and $f(-1) = g(-1)$. It is also obviously a ring homomorphism, since when we add or multiply two polynomials, their respective values at 1 and -1 get multiplied as well. It remains to check that this map is a bijection. To check that it is injective, let us assume that $f(x) + (x^2-1)\mathbb{R}[x]$ and $g(x) + (x^2-1)\mathbb{R}[x]$ are mapped to the same pair (a, b), that is $f(1) = g(1) = a$, $f(-1) = g(-1) = b$. Then $f(x) - g(x)$ has roots 1 and -1, so is divisible by $(x-1)$ and $(x+1)$, hence by $(x-1)(x+1) = x^2-1$. To check that our map is surjective, it is necessary to check that for all pairs a, b there exists a polynomial $f(x)$ with $f(1) = a$, $f(-1) = b$. All polynomials with $f(1) = a$ are of the form $h(x)(x-1)+a$. It is enough to pick $h(x)$ so that $-2h(-1)+a = b$, so $h(-1) = -\frac{1}{2}(b-a)$. For instance, the constant polynomial $h(x) = -\frac{1}{2}(b-a)$ would do. So we have a homomorphism which is injective and surjective, therefore an isomorphism.

3. (**a**) Of course: $(a+I)(b+I)$ is, by definition, $(ab+I)$, which because of commutativity is $(ba+I) = (b+I)(a+I)$.

(**b**) We have $(1+I)(r+I) = (1\cdot r+I) = (r+I) = (r\cdot 1+I) = (r+I)(1+I)$, so $(1+I)$ is a unit of R/I.

4. Let us consider the map $\varphi\colon R \to R$, $\varphi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. This map is a homomorphism: when we add or subtract triangular matrices, their diagonal elements add/subtract, when we multiply triangular matrices, their diagonal elements multiply. The image of φ is clearly S, and the kernel of φ is I, since I consists precisely of triangular matrices with zero diagonal. By First Isomorphism Theorem, S is a subring, I is an ideal, and $S \simeq R/I$. To show that S is not an ideal, note that the product $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not in S.