MA 1112: Linear Algebra II
Tutorial problems, January 29, 2019

1. We begin with computing "convenient" bases of these subspaces. First, we form the transpose matrix of the matrix made of columns of the first system of vectors, and compute its reduced row echelon form:

$$\begin{pmatrix} 0 & 3 & -2 & 2 \\ -9 & 8 & 2 & -3 \\ 4 & 1 & 1 & 1 \end{pmatrix} \overset{(1)\leftrightarrow(3),\frac14(1),(2)+9(1)}{\longmapsto} \begin{pmatrix} 1 & \frac14 & \frac14 & \frac14 \\ 0 & \frac{41}{4} & \frac{17}{4} & -\frac34 \\ 0 & 3 & -2 & 2 \end{pmatrix} \overset{\frac{4}{41}(2),(1)-\frac14(2),(3)-3(2)}{\longmapsto}$$

$$\begin{pmatrix} 1 & 0 & \frac{6}{41} & \frac{11}{41} \\ 0 & 1 & \frac{17}{41} & -\frac{3}{41} \\ 0 & 0 & -\frac{133}{41} & \frac{91}{41} \end{pmatrix} \overset{-\frac{41}{133}(3),(1)-\frac{6}{41}(3),(2)-\frac{17}{41}(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{19} \\ 0 & 1 & 0 & \frac{4}{19} \\ 0 & 0 & 1 & -\frac{13}{19} \end{pmatrix},$$

next we do the same to the second system of vectors:

$$\begin{pmatrix} 6 & 0 & -3 & 1 \\ 3 & 3 & 0 & 5 \\ 9 & -1 & -5 & 0 \end{pmatrix} \overset{\frac16(1),(2)-3(1),(3)-9(1)}{\longmapsto}$$

$$\begin{pmatrix} 1 & 0 & -\frac12 & \frac16 \\ 0 & 3 & \frac32 & \frac92 \\ 0 & -1 & -\frac12 & -\frac32 \end{pmatrix} \overset{\frac13(2),(3)+(2)}{\longmapsto} \begin{pmatrix} 1 & 0 & -\frac12 & \frac16 \\ 0 & 1 & \frac12 & \frac32 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This means that U_1 has a basis consisting of the vectors

$$g_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{7}{19} \end{pmatrix}, g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{4}{19} \end{pmatrix}, g_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{13}{19} \end{pmatrix},$$

and U_2 has a basis consisting of the vectors

$$h_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac12 \\ \frac16 \end{pmatrix}, h_2 = \begin{pmatrix} 0 \\ 1 \\ \frac12 \\ \frac32 \end{pmatrix}.$$

2. The intersection is described by the system of equations

$$a_1g_1 + a_2g_2 + a_3g_3 - b_1h_1 - b_2h_2 = 0.$$

The matrix of this system is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac12 & -\frac12 \\ \frac{7}{19} & \frac{4}{19} & -\frac{13}{19} & -\frac16 & -\frac32 \end{pmatrix}.$$

Let us compute its reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{7}{19} & \frac{4}{19} & -\frac{13}{19} & -\frac{1}{6} & -\frac{3}{2} \end{pmatrix} \overset{(4)-\frac{7}{19}(1),(4)-\frac{4}{19}(2),(4)+\frac{13}{19}(3)}{\longmapsto}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{31}{57} & -\frac{31}{19} \end{pmatrix} \overset{\frac{57}{31}(4),(3)-\frac{1}{2}(4),(1)+(4)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

so b_2 is the only free variable, and once we set $b_2 = t$, we have $b_1 = 3t$, $a_3 = -t$, $a_2 = t$, and $a_1 = 3t$. Therefore, the general vector of the intersection is of the form

$$3t \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{7}{19} \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{4}{19} \end{pmatrix} - t \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{13}{19} \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix},$$

and the intersection is spanned by $\begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix}$.

3. Let us reduce the basis vectors of U_1 using the basis vector of $U_1 \cap U_2$ that we found:

$$\begin{pmatrix} 3 & 1 & -1 & 2 \\ 1 & 0 & 0 & \frac{7}{19} \\ 0 & 1 & 0 & \frac{4}{19} \\ 0 & 0 & 1 & -\frac{13}{19} \end{pmatrix} \overset{\frac{1}{3}(1),(2)-(1)}{\longmapsto} \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{17}{57} \\ 0 & 1 & 0 & \frac{4}{19} \\ 0 & 0 & 1 & -\frac{13}{19} \end{pmatrix}$$

Now, let us find the reduced row echelon form of the resulting matrix:

$$\begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{17}{57} \\ 0 & 1 & 0 & \frac{4}{19} \\ 0 & 0 & 1 & -\frac{13}{19} \end{pmatrix} \overset{-3(1),(2)-(1)}{\longmapsto} \begin{pmatrix} 0 & 1 & -1 & \frac{17}{19} \\ 0 & 0 & 1 & -\frac{13}{19} \\ 0 & 0 & 1 & -\frac{13}{19} \end{pmatrix} \overset{(3)-(2),(1)+(2)}{\longmapsto} \begin{pmatrix} 0 & 1 & -1 & \frac{4}{19} \\ 0 & 0 & 1 & -\frac{13}{19} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We conclude that the vectors

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ \frac{4}{19} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{13}{19} \end{pmatrix}$$

can be chosen for a relative basis.

4. For $U = \text{span}(v_1, v_2)$ to be invariant, it is necessary and sufficient to have $\varphi(v_1), \varphi(v_2) \in U$. Indeed, this condition is necessary because we must have $\varphi(U) \subset U$, and it is sufficient because each vector of U is a linear combination of v_1 and v_2.

We have $\varphi(v_1) = Av_1 = \begin{pmatrix} -3 \\ 8 \\ -8 \end{pmatrix}$ and $\varphi(v_2) = Av_2 = \begin{pmatrix} -1 \\ 8 \\ -8 \end{pmatrix}$. It just remains to see if there are scalars x, y such that $\varphi(v_1) = xv_1 + yv_2$ and scalars z, t such that $\varphi(v_2) = zv_1 + tv_2$. Solving the corresponding systems of linear equations, we see that there are solutions: $\varphi(v_1) = -3v_1 + 5v_2$ and $\varphi(v_2) = -v_1 + 7v_2$. Therefore, this subspace is invariant.