1112: Linear Algebra II
Selected midterm questions from past years

February 13, 2019

1. (a) Show that for every vector $\mathbf{v} \in \mathbb{R}^3$ the map $A_{\mathbf{v}} \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula
$$A_{\mathbf{v}}(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$$
is linear, and show that for $\mathbf{v} \neq 0$ this linear map has rank 2.

   (b) Let U, V and W be three vector spaces. Show that for every two linear operators $A \colon V \to W$ and $B \colon U \to V$ we have
$$\mathrm{rk}(AB) \leqslant \mathrm{rk}(A) \quad \text{and} \quad \mathrm{rk}(AB) \leqslant \mathrm{rk}(B).$$

2. Consider the matrices
$$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) Describe all eigenvalues and eigenvectors of A and B.

   (b) Describe the Jordan normal form of A and find a Jordan basis for A.

3. Assume that for a $n \times n$-matrix A with real matrix elements we have $A^2 = -E$. Prove that $\operatorname{tr} A = 0$.

4. (a) Consider the vector space V of all 2×2-matrices (with obvious addition and multiplication by scalars). Show that for every 2×2-matrix A the map $L_A \colon V \to V$ given by the formula $L_A(X) = AX - XA$, is linear. In the case $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, write down the matrix of L_A relative to the basis $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and compute $\mathrm{rk}(L_A)$.

(b) Let V and W be vector spaces. Show that for every two linear maps $A, B\colon V \to W$ we have

$$\mathrm{rk}(A+B) \leqslant \mathrm{rk}(A) + \mathrm{rk}(B).$$

5. Consider the matrices

$$A = \begin{pmatrix} -2 & -4 & 16 \\ 0 & 2 & 0 \\ -1 & -1 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Describe all eigenvalues and eigenvectors of A and B.

(b) Describe the Jordan normal form of A and find a Jordan basis for A.

6. (a) Show that if for square matrices A and B it is known that A is similar to B, then A^T is similar to B^T (here X^T, as usual, denotes the transpose matrix of X).

(b) Show that (over complex numbers) every square matrix A is similar to A^T.

7. Determine the Jordan normal form and find some Jordan basis for the linear transformation of $\mathbb{R}^3$ that multiplies every vector by the matrix

$$\begin{pmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 2 & -3 & 2 \end{pmatrix}.$$

8. For two $n \times n$-matrices A and B, we have $AB - BA = B$. Show that

(a) $\mathrm{tr}(B) = 0$;

(b) $\mathrm{tr}(B^2) = 0$;

(c) $\mathrm{tr}(B^k) = 0$ for all positive integers k.

9. Define the rank of a linear map. Compute the rank of the linear map $A\colon \mathbb{R}^8 \to \mathbb{R}^4$ whose matrix relative to the standard bases of these spaces is

$$\begin{pmatrix} 3 & 5 & 4 & 2 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 2 & 4 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 & 0 & 2 & -1 \\ -1 & 0 & 1 & 1 & 2 & 1 & 1 & -2 \end{pmatrix}$$

10. In the vector space $V = \mathbb{R}^5$, consider the subspace U spanned by the vectors

$$\begin{pmatrix} 2 \\ 2 \\ 1 \\ 7 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ 1 \\ -12 \\ 6 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Compute $\dim U$.

(b) Which of the vectors $\begin{pmatrix}4\\0\\5\\-3\\-1\end{pmatrix}$, $\begin{pmatrix}2\\1\\8\\4\\2\end{pmatrix}$, $\begin{pmatrix}4\\2\\4\\0\\0\end{pmatrix}$, and $\begin{pmatrix}1\\0\\5\\0\\2\end{pmatrix}$ belong to U?

11. Consider the matrices

$$A = \begin{pmatrix}-3 & 1 & 0\\-1 & -1 & 0\\-1 & -2 & 1\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}-2 & 0 & 1\\0 & 1 & 0\\0 & 0 & -2\end{pmatrix}.$$

Describe the Jordan normal form and find some Jordan basis for A. Do A and B represent the same linear transformation in different coordinate systems? Explain your answer.

12. Which of the maps $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ from the vector space of all polynomials in one variable to the same space are linear? Explain your answers.

$$(\mathcal{A}p)(t) = p(t+1) - p(t),$$
$$(\mathcal{B}p)(t) = p(t)p'(t),$$
$$(\mathcal{C}p)(t) = p(t+1) + p'(t),$$
$$(\mathcal{D}p)(t) = p(t+1) - 1.$$

13. Under what condition a subspace U of a vector space V is said to be an invariant subspace of a linear transformation $A\colon V \to V$? Is the subspace U of $\mathbb{R}^4$ spanned by $\begin{pmatrix}1\\1\\4\\-2\end{pmatrix}$ and $\begin{pmatrix}-2\\-1\\-1\\1\end{pmatrix}$ an invariant subspace of the linear map whose matrix relative to the standard basis of $\mathbb{R}^4$ is

$$\begin{pmatrix}0 & 3 & -3 & -1\\1 & 3 & -1 & 0\\7 & 12 & 2 & 3\\-3 & -6 & 0 & -1\end{pmatrix}?$$

Explain your answer.

14. Determine the Jordan normal form and find some Jordan basis for the matrix $A = \begin{pmatrix}3 & -3 & 1\\2 & -2 & 1\\2 & -3 & 2\end{pmatrix}$. Determine if A and the matrix $B = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ represent the same linear transformation in different coordinate systems.