1112: Linear Algebra II
Selected final exam questions from past years

April 2, 2019

1. (a) Find all eigenvalues and eigenvectors of the matrix
   \[ B = \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix}. \]

   (b) Find the Jordan normal form of the matrix \( B \) from the previous question, and a matrix \( C \) that transforms \( B \) to its Jordan normal form.

   (c) Find a formula for \( B^n \), and use it to find a formula for the \( n \)-th term of the sequence defined recursively by \( a_0 = 2, a_1 = 1, a_{n+1} = -4a_n - 4a_{n-1} \).

2. In the vector space of all polynomials in \( t \) of degree at most 2 with the scalar product
   \[ (p(t), q(t)) = \int_{-1}^{1} p(t)q(t) \, dt, \]
   find the orthogonal basis which is the output of the Gram-Schmidt orthogonalisation applied to the basis \( 2 + 3t, t^2 - 1, t - 1 \).

3. (a) Formulate the Sylvester’s criterion for a quadratic form to be positive definite.

   (b) Determine all values of the parameter \( a \) for which the quadratic form
   \[ q(xe_1 + ye_2 + ze_3) = (18 + a)x^2 + 3y^2 + az^2 + 10xy - (8 + 2a)yz - 4yz \]
   is positive definite.

4. Is the subspace \( U \) of \( \mathbb{R}^4 \) spanned by
   \[ \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \]
   and
   \[ \begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \end{pmatrix} \]
   an invariant subspace of the operator \( A \) whose matrix relative to the standard basis is
   \[ \begin{pmatrix} 0 & 3 & -3 & -1 \\ 1 & 3 & -1 & 0 \\ 7 & 12 & 2 & 3 \\ -3 & -6 & 0 & -1 \end{pmatrix}. \]
   Explain your answer.

5. (a) Find all eigenvalues and eigenvectors of the matrix
   \[ B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}. \]

   (b) Find the Jordan normal form of the matrix \( B \), and a matrix \( C \) which is the transition matrix of some Jordan basis of \( B \).
(c) Find a formula for $B^n$, and use it to find a closed formula for the $n$th terms of the sequences \( \{x_m\}, \{y_m\} \) defined recursively as follows:

\[
\begin{align*}
x_0 &= 1, \quad y_0 = -5, \\
x_{k+1} &= x_k - y_k, \quad y_{k+1} = x_k + 3y_k.
\end{align*}
\]

6. (a) Which bases of a Euclidean space $V$ are called orthogonal? orthonormal?

(b) Show that the $f_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$, and $f_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ form a basis of $\mathbb{R}^3$.

(c) Find the orthogonal basis of $\mathbb{R}^3$ which is the output of the Gram-Schmidt orthogonalisation applied to the basis from the previous question. (The scalar product on the $\mathbb{R}^3$ is the standard one.)

7. (a) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

(b) Consider the vector space $V$ of all polynomials in $t$ of degree at most 2. The bilinear form $\psi_a$ on $V$ (depending on a [real] parameter $a$) is defined by the formula

\[
\psi_a(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)(t-a) \, dt.
\]

Determine all values of $a$ for which $\psi_a$ is positive definite.

8. (a) Determine the Jordan normal form and find some Jordan basis for the matrix

\[
A = \begin{pmatrix} 3 & -4 & 6 \\ 1 & -5 & 3 \\ 0 & -4 & 1 \end{pmatrix}
\]

(b) Find a closed formula for $A^n$.

9. (a) Write down the definition of a Euclidean vector space.

(b) The function $f_a : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ (depending on a real parameter $a$) is defined by the formula

\[
f_a(x_1 e_1 + x_2 e_2 + x_3 e_3, y_1 e_1 + y_2 e_2 + y_3 e_3) = 2x_1 y_1 + (x_1 y_2 + x_2 y_1) + (2a - 1)x_2 y_2 - a(x_1 y_3 + x_3 y_1) - (x_2 y_3 + x_3 y_2) + x_3 y_3
\]

(here $e_1$, $e_2$, $e_3$ is a basis of $\mathbb{R}^3$). Determine all values of $a$ for which $f_a$ is a scalar product.

10. Let $V$ be a vector space. Show that for every two linear operators $A : V \rightarrow V$ and $B : V \rightarrow V$ we have

\[
\text{rk}(AB) \leq \text{rk}(A) \quad \text{and} \quad \text{rk}(AB) \leq \text{rk}(B).
\]

Show that if $B$ is invertible, then $\text{rk}(BA) = \text{rk}(A)$, and give an example showing that this equality might hold even if $B$ is not invertible.

11. (a) Determine the Jordan normal form and find some Jordan basis for the matrix

\[
A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix}
\]

(b) Find a closed formula for $A^n$. 
12. (a) A quadratic form $Q$ on the space $\mathbb{R}^3$ is defined by the formula
\[
Q(xe_1 + ye_2 + ze_3) = (20 + 4a)x^2 + 12(1 + a)xz + 6y^2 + 3z^2.
\]
Find all values of the parameter $a$ for which this form is positive definite.

13. A square matrix $A$ (of some size $n \times n$) satisfies the condition
\[
A^2 - 8A + 15I = 0.
\]
(a) Show that this matrix is similar to a diagonal matrix.
(b) Show that for every positive integer $k \geq 8$ there exists a matrix $A$ satisfying the above condition with $\text{tr}(A) = k$.

14. (a) Determine the Jordan normal form and find some Jordan basis for the matrix
\[
A = \begin{pmatrix} 2 & -5 & 3 \\ 2 & -6 & 4 \\ 3 & -9 & 6 \end{pmatrix}.
\]
(b) Find a closed formula for $A^n$.

15. (a) A quadratic form $Q$ on the three-dimensional space with a basis $e_1, e_2, e_3$ is defined by the formula
\[
Q(xe_1 + ye_2 + ze_3) = 3x^2 + 2axy + (2 - 2a)xz + (a + 2)y^2 + 2ayz + 3z^2
\]
Find all values of the parameter $a$ for which this form is positive definite.

16. In the vector space $V = \mathbb{R}^5$, consider the subspace $U$ spanned by the vectors
\[
\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ -12 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}
\]
(a) Compute $\dim U$.
(b) Which of the vectors $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ belong to $U$?

17. Consider the matrices
\[
A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
(a) Describe the Jordan normal form and find some Jordan basis for $A$.
(b) Is $A$ similar to $B$? Is $A^2$ similar to $B$? Explain your answers.

18. Consider the vector space $V$ of all $n \times n$-matrices, and define a bilinear form on this space by the formula $(A, B) = \text{tr}(AB^T)$.
(a) Show that this bilinear form is a scalar product on the space of all matrices.
(b) Show that with respect to that scalar product the subspace of all symmetric matrices (matrices $A$ with $A = A^T$) is the orthogonal complement of the space of all skew-symmetric matrices (matrices $A$ with $A = -A^T$).

19. Does there exist a $9 \times 9$-matrix $B$ for which the matrix $B^2$ has the Jordan normal form with blocks of sizes 4,3,2 appearing once, each block with the eigenvalue 0? Same question for the block sizes 4,4,1.