Lemma 2. The vectors \( \varphi^k \) that vanish, so that \( \varphi^k \) argument that we had for \( k = 2 \) to this general case. We assume that \( k \) is actually the smallest power of \( \varphi \) that vanishes, so that \( \varphi^{k-1} \neq 0 \).

Let us put, for each \( p \), \( N_p = \ker(\varphi^p) \). Of course, we have \( N_k = N_{k+1} = N_{k+2} = \ldots = V \).

We shall now construct a basis of \( V \) of a very particular form. It will be constructed in \( k \) steps. First, we find a basis of \( V = N_k \) relative to \( N_{k-1} \). Let \( e_1, \ldots, e_s \) be vectors of this basis.

The following result is proved in the same way as the one from the previous class:

**Lemma 1.** The vectors \( e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s) \) are linearly independent relative to \( N_{k-2} \).

**Proof.** Indeed, assume that \( a_1e_1 + \ldots + a_se_s + b_1\varphi(e_1) + \ldots + b_s\varphi(e_s) \in N_{k-2} \).

Since \( e_i \in N_k \), we have \( \varphi(e_i) \in N_{k-1} \), so

\[
\begin{align*}
a_1e_1 + \ldots + a_se_s & = -b_1\varphi(e_1) - \ldots - b_s\varphi(e_s) + N_{k-2} \subset N_{k-1},
\end{align*}
\]

which means that \( a_1 = \ldots = a_s = 0 \). Thus,

\[
\varphi(b_1e_1 + \ldots + b_se_s) = b_1\varphi(e_1) + \ldots + b_s\varphi(e_s) \in N_{k-2},
\]

so \( b_1e_1 + \ldots + b_se_s \in N_{k-1} \), and we deduce that \( b_1 = \ldots = b_s = 0 \), thus the lemma follows. \( \square \)

Now we find vectors \( f_1, \ldots, f_t \) which form a basis of \( N_{k-1} \) relative to \( \text{span}(\varphi(e_1), \ldots, \varphi(e_s)) + N_{k-2} \). Absolutely analogously one can prove

**Lemma 2.** The vectors \( e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s), \varphi^2(e_1), \ldots, \varphi^2(e_s), f_1, \ldots, f_t, \varphi(f_1), \ldots, \varphi(f_t) \) are linearly independent relative to \( N_{k-3} \).

**Proof.** Let us assume that

\[
\begin{align*}
a_1^{(1)}e_1 + \ldots + a_s^{(1)}e_s + a_1^{(2)}\varphi(e_1) + \ldots + a_s^{(2)}\varphi(e_s) + a_1^{(3)}\varphi^2(e_1) + \ldots + a_s^{(3)}\varphi^2(e_s) + \\
b_1^{(1)}f_1 + \ldots + b_t^{(1)}f_t + b_1^{(2)}\varphi(f_1) + \ldots + b_t^{(2)}\varphi(f_t) & \in N_{k-3}.
\end{align*}
\]

We note that

\[
\begin{align*}
a_1^{(2)}\varphi(e_1) + \ldots + a_s^{(2)}\varphi(e_s) & \in N_{k-1}, \\
a_1^{(3)}\varphi^2(e_1) + \ldots + a_s^{(3)}\varphi^2(e_s) & \in N_{k-2} \\
b_1^{(1)}f_1 + \ldots + b_t^{(1)}f_t & \in N_{k-1}, \\
b_1^{(2)}\varphi(f_1) + \ldots + b_t^{(2)}\varphi(f_t) & \in N_{k-2},
\end{align*}
\]
so we have $a_1^{(1)} e_1 + \ldots + a_s^{(1)} e_s \in N_{k-1}$, and hence $a_1^{(1)} = a_s^{(1)} = 0$ since the vectors $e_1, \ldots, e_s$ form a relative basis. Thus, we have

$$a_1^{(2)} \varphi(e_1) + \ldots + a_s^{(2)} \varphi(e_s) + a_1^{(3)} \varphi^2(e_1) + \ldots + a_s^{(3)} \varphi^2(e_s) + b_1^{(1)} f_1 + \ldots + b_t^{(1)} f_t + b_1^{(2)} \varphi(f_1) + \ldots + b_1^{(2)} \varphi(f_1) \in N_{k-3}.$$  

Now,

$$a_1^{(2)} \varphi(e_1) + \ldots + a_s^{(2)} \varphi(e_s) \in \text{span}(\varphi(e_1), \ldots, \varphi(e_s)),
\quad a_1^{(3)} \varphi^2(e_1) + \ldots + a_s^{(3)} \varphi^2(e_s) \in N_{k-2},
\quad b_1^{(2)} \varphi(f_1) + \ldots + b_t^{(2)} \varphi(f_t) \in N_{k-2},$$

so $b_1^{(1)} f_1 + \ldots + b_t^{(1)} f_t \in \text{span}(\varphi(e_1), \ldots, \varphi(e_s)) + N_{k-2}$, and hence $b_1^{(1)} = b_t^{(1)} = 0$ since the vectors $f_1, \ldots, f_t$ form a relative basis. Our original assumption simplifies to

$$a_1^{(2)} \varphi(e_1) + \ldots + a_s^{(2)} \varphi(e_s) + a_1^{(3)} \varphi^2(e_1) + \ldots + a_s^{(3)} \varphi^2(e_s) + b_1^{(2)} f_1 + \ldots + b_1^{(2)} f_t \in N_{k-3},$$

which can be rewritten as

$$\varphi(a_1^{(2)} e_1 + \ldots + a_s^{(2)} e_s + a_1^{(3)} \varphi(e_1) + \ldots + a_s^{(3)} \varphi(e_s) + b_1^{(2)} f_1 + \ldots + b_1^{(2)} f_t) \in N_{k-3},$$

implying

$$a_1^{(2)} e_1 + \ldots + a_s^{(2)} e_s + a_1^{(3)} \varphi(e_1) + \ldots + a_s^{(3)} \varphi(e_s) + b_1^{(2)} f_1 + \ldots + b_1^{(2)} f_t \in N_{k-2},$$

which by the previous lemma and the relative basis property of $f_1, \ldots, f_t$ shows that all the coefficients are equal to zero. \qed

Next we find a basis of $N_{k-2}$ relative to $\text{span}(\varphi^2(e_1), \ldots, \varphi^2(e_s), \varphi(f_1), \ldots, \varphi(f_t)) + N_{k-3}$, etc. We continue that extension process until we end up with a basis of $V$ of the following form:

$$e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s), \varphi^2(e_1), \ldots, \varphi^2(e_s), \varphi^{k-1}(e_1), \ldots, \varphi^{k-1}(e_s),
\quad f_1, \ldots, f_t, \varphi(f_1), \ldots, \varphi^{k-2}(f_1), \varphi^{k-2}(f_t),
\quad \ldots,
\quad h_1, \ldots, h_p,$$

where the first line contains several “threads” $e_1, \varphi(e_1), \ldots, \varphi^{k-1}(e_1)$ of length $k$, the second line — several threads of length $k-1, \ldots$, the last line — several threads of length 1, that is several vectors from $N_1$.

Let us rearrange the basis vectors so that vectors forming a thread are all next to each other:

$$e_1, \varphi(e_1), \ldots, \varphi^{k-1}(e_1), e_s, \varphi(e_s), \ldots, \varphi^{k-1}(e_s),
\quad f_1, \varphi(f_1), \ldots, \varphi^{k-2}(f_1), f_t, \varphi(f_t), \ldots, \varphi^{k-2}(f_t),
\quad \ldots,
\quad g_1, \ldots, g_u.$$

Relative to that basis, the linear transformation $\varphi$ has the matrix made of Jordan blocks

$$J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & 0 & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix},$$

one block $J_1$ for each thread of length 1.