Another important calculation which we shall be doing quite a bit in the following classes is computing a basis of a vector space relative to its subspace. (Once again, we assume the spaces presented as linear spans of several vectors.)

The general set-up here is as follows. We have the ambient vector space $V$, inside it a subspace $W = \text{span}(e_1, \ldots, e_k)$, and then a subspace $W' = \text{span}(f_1, \ldots, f_l)$ of $W$. In this case, it is reasonable to bring the matrix of vectors spanning $W'$ to its reduced column echelon form, and then reduce the matrix of vectors spanning $W$ with respect to the thus obtained reduced column echelon matrix using the pivots of the latter. The resulting matrix then should be brought to its reduced column echelon form, giving a relative basis.

**Example 1.** Consider the subspace $W$ of $\mathbb{R}^5$ equal to the space $U_2$ from the previous class, that is the span of the vectors
\[
\begin{pmatrix}
2 \\ 1 \\ 0 \\ 1 \\ 1
\end{pmatrix}, \quad \begin{pmatrix}
2 \\ -1 \\ -2 \\ -3 \\ -1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\ -3 \\ -2 \\ 2 \\ 1
\end{pmatrix}.
\]
Let us also define the subspace $W'$ as the span of the vectors
\[
\begin{pmatrix}
1 \\ -1 \\ 3 \\ 2 \\ -7
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
3 \\ 1 \\ 1 \\ 2 \\ -7
\end{pmatrix}.
\]

Let us first find a “convenient” basis of $W'$. Using transpose matrices again, we perform the row operations
\[
\begin{pmatrix}
1 \\ -1 \\ 3 \\ 2 \\ -7
\end{pmatrix} \xrightarrow{(2) \to 3(1), 1/4(2)} \begin{pmatrix}
1 \\ -1 \\ 3 \\ 2 \\ -7
\end{pmatrix} \xrightarrow{(1) \to (1) + (2)} \begin{pmatrix}
1 \\ 0 \\ 1 \\ 1 \\ -2
\end{pmatrix}.
\]

Recall that the basis of $W$ is given by the transpose of the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -2/3 \\
0 & 1 & 0 & 1 & 7/3 \\
0 & 0 & 1 & 1 & -4/3
\end{pmatrix}.
\]

From this, it is already clear that the rows of the former matrix are $r_1 + r_3$ and $r_2 - 2r_3$, where $r_i$ are the rows of the latter matrix, so $W'$ is indeed a subspace of $W$. Let us now reduce rows of $W$ with respect to rows of $W'$:
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -2/3 \\
0 & 1 & 0 & 1 & 7/3 \\
0 & 0 & 1 & 1 & -4/3
\end{pmatrix} \xrightarrow{(1) \to (1) + (2) - (2')} \begin{pmatrix}
0 & 0 & -1 & -1 & 4/3 \\
0 & 0 & 2 & 2 & -8/3 \\
0 & 0 & 1 & 1 & -4/3
\end{pmatrix}.
\]

Clearly, the reduced row echelon form of this matrix is
\[
\begin{pmatrix}
0 & 0 & 1 & 1 & -4/3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
so the vector

\[ v = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -4/3 \end{pmatrix} \]

can be chosen to form the relative basis, that is a set of linearly independent vectors that, together with a basis of \( W' \), give us a basis of \( W \).

**Invariant subspaces**

Our next step is to introduce a yet another definition that will be needed to study arbitrary linear transformations, that of an invariant subspace.

**Definition 1.** Let \( V \) be a vector space, and \( \varphi: V \to V \) be a linear transformation. A subspace \( U \) of \( V \) is said to be *invariant* under \( \varphi \) if \( \varphi(U) \subset (U) \), that is \( \varphi(u) \in U \) for all \( u \in U \).

**Example 2.** All multiples of an eigenvector of \( \varphi \) form a subspace of \( V \) that is invariant under \( \varphi \). Indeed, all multiples of any vector form a subspace, and if it is an eigenvector, then \( \varphi \) maps any vector from this subspace to its multiple.

Let us use this opportunity to fix some notation related to eigenvectors. Recall that eigenvalues of a linear transformation \( \varphi \) of an \( n \)-dimensional space \( V \) are roots of \( \det(A_{\varphi,e} - tI_n) \), where \( e_1, \ldots, e_n \) is any basis of \( V \).

**Definition 2.** The expression \( \det(A_{\varphi,e} - tI_n) \) is called the *characteristic polynomial* of the linear transformation \( \varphi \). It is often denoted \( \chi_{\varphi}(t) \).

By inspection, \( \chi_{\varphi}(t) \) is a polynomial in \( t \) of degree \( n \) with leading coefficient \( (-1)^n \). Note that over complex numbers every polynomial has a root, and so every linear transformation has an eigenvector.

The example of eigenvectors is, in a sense, a very useful motivation for introducing invariant subspaces. Namely, suppose that \( U \subset V \) is an invariant subspace of a linear transformation \( \varphi \). Let \( e_1, \ldots, e_k \) be a basis of \( U \) and \( f_1, \ldots, f_l \) a basis of \( V \) relative to \( U \), so that \( e_1, \ldots, e_k, f_1, \ldots, f_l \) is a basis of \( V \). Then, by direct inspection, the matrix of the linear transformation \( \varphi \) with respect to this basis has the block-triangular form

\[ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \]

where \( A \) is the matrix describing how \( \varphi \) transforms the invariant subspace \( U \). Our hunt for invariant subspaces is ultimately motivated by a wish to reduce a “big” problem of working with an arbitrary linear transformations to similar but “smaller” ones.

From now on, we shall work with complex numbers as scalars for a while, thus ensuring that linear transformations have many invariant subspaces. This is not true over real numbers: some linear transformations (e.g. rotations in 2D) have no nontrivial invariant subspaces at all.