MA 1112: Linear Algebra II
Homework problems for February 11, 2019

Solutions to this problem sheet are to be handed in after our class at 11am on Monday. Please attach a cover sheet with a declaration `http://tcd-ie.libguides.com/plagiarism/declaration` confirming that you know and understand College rules on plagiarism. On the same cover sheet, please put your name, student number, and name of the degree (Maths/TP/TSM), and staple all the sheets together. (Failure to do that may result in misplaced/lost sheets, for which no responsibility can be taken by instructors.)

Recall that in class we proved that for every linear transformation $\varphi\colon V \to V$ with $\varphi^k = 0$ for some k, it is possible to choose a basis

$$\begin{gathered} e_1^{(1)}, e_2^{(1)}, e_3^{(1)}, \ldots, e_{n_1}^{(1)}, \\ e_1^{(2)}, e_2^{(2)}, \ldots, e_{n_2}^{(2)}, \\ \ldots \\ e_1^{(l)}, \ldots, e_{n_l}^{(l)} \end{gathered}$$

of V such that for each "thread"

$$e_1^{(p)}, e_2^{(p)}, \ldots, e_{n_p}^{(p)}$$

we have

$$\varphi(e_1^{(p)}) = e_2^{(p)}, \varphi(e_2^{(p)}) = e_3^{(p)}, \ldots, \varphi(e_{n_p}^{(p)}) = 0.$$

In questions 1–4, your goal is, given a vector space V and a transformation $\varphi\colon V \to V$,

- compute φ^2, φ^3, ..., and find the smallest k such that $\varphi^k = 0$,
- compute $\operatorname{rk}(\varphi)$, $\operatorname{rk}(\varphi^2)$, ..., and $\operatorname{null}(\varphi)$, $\operatorname{null}(\varphi^2)$, ...,
- find some basis of V split into several "threads" on which φ acts as described above.

1. $V = \mathbb{R}^2$, φ is multiplication by the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

2. $V = \mathbb{R}^3$, φ is multiplication by the matrix $A = \begin{pmatrix} -18 & 24 & 36 \\ -18 & 24 & 36 \\ 3 & -4 & -6 \end{pmatrix}$.

3. $V = \mathbb{R}^3$, φ is multiplication by the matrix $A = \begin{pmatrix} -36 & -24 & 2 \\ 54 & 36 & -3 \\ 3 & 2 & 0 \end{pmatrix}$.

4. $V = \mathbb{R}^4$, φ is multiplication by the matrix $A = \begin{pmatrix} 2 & 4 & 4 & 4 \\ -2 & -4 & -5 & -6 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix}$.

5. For an $n \times n$-matrix A we have $A^N = 0$ for some N. Prove that $A^n = 0$.