MA 1112: Linear Algebra II  
Homework problems for January 28, 2019

Solutions to this problem sheet are to be handed in after our class at 11am on Monday. Please attach a cover sheet with a declaration [http://tcd-ie.libguides.com/plagiarism/declaration](http://tcd-ie.libguides.com/plagiarism/declaration) confirming that you know and understand College rules on plagiarism. On the same cover sheet, please put your name, student number, and name of the degree (Maths/TP/TSM), and staple all the sheets together. (Failure to do that may result in misplaced/lost sheets, for which no responsibility can be taken by instructors.)

1. For each of the following matrices $A$, viewed as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$,
   - compute the rank of $A$;
   - describe all eigenvalues and eigenvectors of $A$;
   - determine whether there exists a change of coordinates making the matrix $A$ diagonal.

   (a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; (b) $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & -1 & 4 \end{pmatrix}$; (c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{pmatrix}$.

2. Compute the rank of the matrix

   \[
   \begin{pmatrix}
   1 & 1 & 0 & 0 & 0 & 0 \\
   0 & 1 & 1 & 0 & 0 & 0 \\
   0 & 0 & 1 & 1 & 0 & 0 \\
   0 & 0 & 0 & 1 & 1 & 0 \\
   0 & 0 & 0 & 0 & 1 & 1 \\
   1 & 0 & 0 & 0 & 0 & 1
   \end{pmatrix}
   \]

3. Let $V$ be the space of all polynomials in $t$ of degree at most $n$, and let $\varphi: V \to V$ be the linear transformation given by $(\varphi(p(t)) = p'(t) + 3p''(t)$. Find the eigenvalues and the eigenvectors of $\varphi$, and show that there is no basis of $V$ relative to which the matrix of $\varphi$ is diagonal. (Hint: first show that $\varphi$ has no non-zero eigenvalues.)

4. Prove that for every $n \times m$-matrix $A$ of rank $1$ there exist an $n \times 1$-matrix $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ and an $1 \times n$-matrix $C = (c_1, c_2, \ldots, c_m)$ such that

   \[
   A = BC = \begin{pmatrix}
   b_1 c_1 & b_1 c_2 & \ldots & b_1 c_m \\
   b_2 c_1 & b_2 c_2 & \ldots & b_2 c_m \\
   \vdots & \vdots & \ddots & \vdots \\
   b_n c_1 & b_n c_2 & \ldots & b_n c_m
   \end{pmatrix},
   \]

   (Hint: denote by $B$ a nonzero column of the matrix $A$, and find an appropriate $C$.)

5. Let $\alpha: U \to V$ and $\beta: V \to W$ be linear maps of vector spaces. Prove that $\text{rk}(\beta \circ \alpha) \leq \text{rk}(\alpha)$ and $\text{rk}(\beta \circ \alpha) \leq \text{rk}(\beta)$. (As always, $\beta \circ \alpha$ denotes the composite map $(\beta \circ \alpha)(u) = \beta(\alpha(u))$.)