MA 1112: Linear Algebra II
Selected answers/solutions to the assignment for February 11, 2019

1. In this case, $\varphi^2 = 0$, $\mathrm{rk}(\varphi) = 1$, $\mathrm{rk}(\varphi^k) = 0$ for $k \geqslant 2$, $\mathrm{null}(\varphi) = 1$, $\mathrm{null}(\varphi^k) = 2$ for $k \geqslant 2$. Moreover, $\mathrm{Ker}(\varphi) = \{\begin{pmatrix} t \\ -t \end{pmatrix}\}$.

We have a sequence of subspaces $V = \mathrm{Ker}\,\varphi^2 \supset \mathrm{Ker}\,\varphi \supset \{0\}$. The first one relative to the second one is one-dimensional (since $\mathrm{null}\,\varphi^2 - \mathrm{null}\,\varphi = 1$). Putting $t = 1$ in the formula above, we get the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ that forms a basis of the kernel, and for the relative basis we can take the basis vector of $\mathbb{R}^2$ making up for the missing pivot, that is $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This vector gives rise to a thread $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \varphi(f) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ of length 2. Since our space is 2-dimensional, this thread forms a basis.

2. In this case, $\varphi^2 = 0$, $\mathrm{rk}\,\varphi = 1$, $\mathrm{rk}\,\varphi^k = 0$ for $k \geqslant 2$, $\mathrm{null}(\varphi) = 2$, $\mathrm{null}(\varphi^k) = 3$ for $k \geqslant 2$. Moreover, $\mathrm{Ker}(\varphi) = \{\begin{pmatrix} \frac{4s+6t}{3} \\ s \\ t \end{pmatrix}\}$.

We have a sequence of subspaces $V = \mathrm{Ker}\,\varphi^2 \supset \mathrm{Ker}\,\varphi \supset \{0\}$. The first one relative to the second one is one-dimensional (since $\mathrm{null}\,\varphi^2 - \mathrm{null}\,\varphi = 1$). The kernel of φ has a basis $\begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 6/3 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the values $s = 1, t = 0$ and $s = 0, t = 1$ of the free variables), and after computing the reduced column echelon form, we see that for a relative basis we may take the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This vector gives rise to a thread $f, \varphi(f) = \begin{pmatrix} 36 \\ 36 \\ -6 \end{pmatrix}$. It remains to find a basis of $\mathrm{Ker}(\varphi)$ relative to the span of $\varphi(f)$. Column reduction of the basis of $\mathrm{Ker}(\varphi)$ by $\varphi(f)$ leaves us with the vector $g = \begin{pmatrix} 0 \\ 1 \\ -2/3 \end{pmatrix}$. Overall, the vectors $f, \varphi(f), g$ form a basis of V consisting of two threads, one of length 2 $(f, \varphi(f))$ and the other one of length 1 (g).

3. In this case, $\varphi^2 = \begin{pmatrix} 6 & 4 & 0 \\ -9 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\varphi^3 = 0$, $\mathrm{rk}\,\varphi = 2$, $\mathrm{rk}\,\varphi^2 = 1$, $\mathrm{rk}\,\varphi^k = 0$ for $k \geqslant 3$, $\mathrm{null}(\varphi) = 1$, $\mathrm{null}(\varphi^2) = 2$, $\mathrm{null}(\varphi^k) = 3$ for $k \geqslant 3$.

We have a sequence of subspaces $V = \mathrm{Ker}\,\varphi^3 \supset \mathrm{Ker}\,\varphi^2 \supset \mathrm{Ker}\,\varphi \supset \{0\}$. The first one relative to the second one is one-dimensional ($\mathrm{null}\,\varphi^3 - \mathrm{null}\,\varphi^2 = 1$). We have $\mathrm{Ker}(\varphi^2) = \begin{pmatrix} -2/3s \\ s \\ t \end{pmatrix}$, so it has a basis of vectors $\begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the value $s = 1, t = 0$ and $s = 0, t = 1$ of the free variables respectively), and after computing the reduced column echelon form we see that for a relative basis we may take the vector $f = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. We have $\varphi(f) = \begin{pmatrix} -24 \\ 36 \\ 2 \end{pmatrix}$, $\varphi^2(f) = \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix}$, and this thread of length 3 forms a basis of our three-dimensional space V.

4. In this case, $\varphi^2 = 0$, $\mathrm{rk}(\varphi) = 2$, $\mathrm{rk}(\varphi^k) = 0$ for $k \geqslant 2$, $\dim \mathrm{Ker}(\varphi) = 2$, $\dim \mathrm{Ker}(\varphi^k) = 4$ for

$k \geqslant 2$. Moreover, $\mathrm{Ker}(\varphi) = \{ \begin{pmatrix} -2s \\ s \\ -2t \\ t \end{pmatrix} \}$.

We have a sequence of subspaces $V = \mathrm{Ker}(\varphi^2) \supset \mathrm{Ker}(\varphi) \supset \{0\}$. The first one relative to the second one is two-dimensional ($\mathrm{null}(\varphi^2) - \mathrm{null}(\varphi) = 2$). Clearly, the vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$ (corresponding to the values $s = 1, t = 0$ and $s = 0, t = 1$ of the free variables respectively) form a basis of $\mathrm{Ker}\,\varphi$, and after computing the reduced column echelon form we see that for a relative basis we may take the vectors $f_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors give rise to threads f_1, $\varphi(f_1) = \begin{pmatrix} 4 \\ -4 \\ 4 \\ -2 \end{pmatrix}$ and f_2, $\varphi(f_2) = \begin{pmatrix} 4 \\ -6 \\ 8 \\ -4 \end{pmatrix}$. These two threads together contain four vectors, and we have a basis.

5. The transformation φ that multiplies every vector by A satisfies $\varphi^N = 0$, so the result from class applies; let us find a basis for φ consisting of several threads. Each thread is of length at most n, since a basis consists of n vectors altogether. Clearly, all vectors from a thread of length l are mapped to zero by φ^l: the last vector is mapped to zero by φ, the previous one — by φ^2 (since φ maps it to the last one, and then one more application of φ maps it to zero), etc. It follows that every individual basis vector is mapped to zero by φ^n, and so is every their combination — it follows that all vectors are mapped to 0, so $A^n = 0$.