MA 1112: Linear Algebra II
Selected answers/solutions to the assignment for February 4

1. The reduced column echelon form of the matrix whose columns are the spanning vectors of U_1 is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 6 & 0 \\ -7 & 9 & 0 \end{pmatrix},$$

and its first two nonzero columns can be taken as a basis.

The reduced column echelon form of the matrix whose columns are the spanning vectors of U_2 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{16}{5} & \frac{17}{5} & -\frac{4}{5} & 0 \end{pmatrix},$$

and its first three nonzero columns can be taken as a basis.

2. The intersection is described by the system of equations

$$a_1g_1 + a_2g_2 - b_1h_1 - b_2h_2 - b_3h_3 = 0,$$

where we denote by g_1, g_2 the basis of U_1 we found, and by h_1, h_2, h_3 the basis of U_2.The matrix of this system of equations is

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -5 & 6 & 0 & 0 & -1 \\ -7 & 9 & \frac{16}{5} & -\frac{17}{5} & \frac{4}{5} \end{pmatrix}$$

and it reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{3}{2} \end{pmatrix}$$

so b_3 is a free variable. Setting $b_3 = t$, we obtain $a_1 = -2t$, $a_2 = -\frac{3}{2}t$. The corresponding basis vector $a_1g_1 + a_2g_2$ is $\begin{pmatrix} -2 \\ -\frac{3}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$.

3. Reducing the basis vectors of U_1 using the basis vector we just found, we obtain the matrix

$$\begin{pmatrix} 0 & 0 \\ -\frac{3}{4} & 1 \\ -\frac{9}{2} & 6 \\ -\frac{27}{4} & 9 \end{pmatrix},$$

and the reduced column echelon form of this matrix is easily seen to be

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 6 & 0 \\ 9 & 0 \end{pmatrix},$$

so the vector $\begin{pmatrix} 0 \\ 1 \\ 6 \\ 9 \end{pmatrix}$ can be taken as the relative basis vector.

4. Reducing the basis vectors of U_2 using the basis vector we found, we obtain the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ -\frac{59}{20} & \frac{17}{5} & -\frac{4}{5} \end{pmatrix},$$

and the reduced column echelon form of this matrix is manifestly

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{17}{5} & -\frac{4}{5} & 0 \end{pmatrix},$$

so its two nonzero columns can be taken as the relative basis.

5. For $U = \mathrm{span}(v_1, v_2)$ to be invariant, it is necessary and sufficient to have $\varphi(v_1), \varphi(v_2) \in U$. Indeed, this condition is necessary because we must have $\varphi(U) \subset U$, and it is sufficient because each vector of U is a linear combination of v_1 and v_2.

We have $\varphi(v_1) = Av_1 = \begin{pmatrix} -5 \\ 10 \\ -13 \end{pmatrix}$ and $\varphi(v_2) = Av_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$. It just remains to see if there are scalars x, y such that $\varphi(v_1) = xv_1 + yv_2$ and scalars z, t such that $\varphi(v_2) = zv_1 + tv_2$. Solving the corresponding systems of linear equations, we see that there are solutions: $\varphi(v_1) = 2v_1 - 3v_2$ and $\varphi(v_2) = 3v_1 + 2v_2$. Therefore, this subspace is invariant.