1. The reduced column echelon form of the matrix whose columns are the spanning vectors of $U_1$ is
\[
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
-5 & 6 & 0 & 0 & 1 \\
-7 & 9 & 0 & 0 & 0
\end{pmatrix},
\]
and its first two nonzero columns can be taken as a basis.

The reduced column echelon form of the matrix whose columns are the spanning vectors of $U_2$ is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-16 & -17 & -4 & 0 & 0
\end{pmatrix},
\]
and its first three nonzero columns can be taken as a basis.

2. The intersection is described by the system of equations
\[
a_1g_1 + a_2g_2 - b_1h_1 - b_2h_2 - b_3h_3 = 0,
\]
where we denote by $g_1, g_2$ the basis of $U_1$ we found, and by $h_1, h_2, h_3$ the basis of $U_2$. The matrix of this system of equations is
\[
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
-5 & 6 & 0 & 0 & 1 \\
-7 & 9 & 16 & 17 & 4
\end{pmatrix}
\]
and it reduced row echelon form is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & \frac{3}{2} \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & \frac{3}{2}
\end{pmatrix}
\]
so $b_3$ is a free variable. Setting $b_3 = t$, we obtain $a_1 = -2t$, $a_2 = -\frac{3}{2}t$. The corresponding basis vector $a_1g_1 + a_2g_2$ is \[
\begin{pmatrix}
-2 \\
-\frac{3}{2} \\
1 \\
\frac{3}{2}
\end{pmatrix}.
\]

3. Reducing the basis vectors of $U_1$ using the basis vector we just found, we obtain the matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & 1 & 0 & 6 & 9
\end{pmatrix},
\]
and the reduced column echelon form of this matrix is easily seen to be
\[
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
6 & 0 \\
9 & 0
\end{pmatrix}.
\]
so the vector \( \begin{pmatrix} 0 \\ 1 \\ 6 \\ 9 \end{pmatrix} \) can be taken as the relative basis vector.

4. Reducing the basis vectors of \( \mathbf{U}_2 \) using the basis vector we found, we obtain the matrix
\[
\begin{pmatrix}
0 & 0 & 0 \\
-\frac{3}{4} & 1 & 0 \\
\frac{1}{4} & 0 & 1 \\
-\frac{179}{20} & \frac{17}{5} & -\frac{4}{5}
\end{pmatrix},
\]
and the reduced column echelon form of this matrix is manifestly
\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{17}{5} & -\frac{4}{5} & 0
\end{pmatrix},
\]
so its two nonzero columns can be taken as the relative basis.

5. For \( \mathbf{U} = \text{span}(v_1, v_2) \) to be invariant, it is necessary and sufficient to have \( \varphi(v_1), \varphi(v_2) \in \mathbf{U} \). Indeed, this condition is necessary because we must have \( \varphi(\mathbf{U}) \subset \mathbf{U} \), and it is sufficient because each vector of \( \mathbf{U} \) is a linear combination of \( v_1 \) and \( v_2 \).

We have \( \varphi(v_1) = A v_1 = \begin{pmatrix} -5 \\ 10 \\ -13 \end{pmatrix} \) and \( \varphi(v_2) = A v_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \). It just remains to see if there are scalars \( x, y \) such that \( \varphi(v_1) = x v_1 + y v_2 \) and scalars \( z, t \) such that \( \varphi(v_2) = z v_1 + t v_2 \). Solving the corresponding systems of linear equations, we see that there are solutions: \( \varphi(v_1) = 2 v_1 - 3 v_2 \) and \( \varphi(v_2) = 3 v_1 + 2 v_2 \). Therefore, this subspace is invariant.