MA 1112: Linear Algebra II

Selected answers/solutions to the assignment for January 28

1. (**a**) $\text{rk}(A) = 1$ (all columns are the same, so there is just one linearly independent column), eigenvectors are 0 and 3, there are two linearly independent eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ for the first of them (and every eigenvector is their linear combination), and every eigenvector for the second of them is proportional to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Since there are three linearly independent eigenvectors, there is a change of coordinates making this matrix diagonal.

(**b**) $\text{rk}(A) = 3$, eigenvectors are 2 and 3, every eigenvector for the first one is proportional to $\begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$, every eigenvector for the second one is proportional to $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Since we do not have three linearly independent eigenvectors, there is no change of coordinates making this matrix diagonal.

(**c**) $rk(A) = 3$, eigenvectors are -2, 1, and 3, every eigenvector for the first one is proportional to $\begin{pmatrix} 1/4 \\ -1/2 \\ 1 \end{pmatrix}$, every eigenvector for the second one is proportional to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, every eigenvector for the third one is proportional to $\begin{pmatrix} 1/9 \\ 1/3 \\ 1 \end{pmatrix}$. There are three linearly independent eigenvectors, so there is a change of coordinates making this matrix diagonal.

2. The kernel of our matrix consists of vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$ for which $x_1 = -x_2$, $x_2 = -x_3$, $x_3 = -x_4$, $x_4 = -x_5$, $x_5 = -x_6$, $x_6 = -x_1$. Overall, all vectors in the kernel are proportional to $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, so the nullity of A is 1, and $\mathrm{rk}(A) = \dim(V) - 1 = 5$.

3. Suppose that $p(t)$ is an eigenvector, so that $\varphi(p(t)) = cp(t)$. Note that $\varphi(p(t)) = p'(t) + 3p''(t)$ is a polynomial of degree less than the degree of $p(t)$, so it can be proportional to $p(t)$ only if it is equal to zero. Thus, the only eigenvalue is zero. Also, if $p'(t) + 3p''(t) = 0$, we see that $(p(t) + 3p'(t))' = 0$, so $p(t) + 3p'(t)$ is a constant polynomial. Since $3p'(t)$ is of degree less than the degree of $p(t)$, this means that $p(t)$ must be a constant polynomial itself, or else its leading term will not cancel. Thus, φ has only one eigenvector, so does not admit a basis of eigenvectors, and therefore there is no basis of V relative to which the matrix of φ is diagonal.

4. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be columns of A. Since $\mathrm{rk}(A) = 1$, there is at least one nonzero column. Assume that $\mathbf{a}_l \neq 0$. For every $k \neq l$, the column $\mathbf{a}_k$ has to be linearly dependent with $\mathbf{a}_l$ because the rank of our matrix is 1: $x_k\mathbf{a}_k + y_k\mathbf{a}_l = 0$. Since $\mathbf{a}_l \neq 0$, we have $x_k \neq 0$ (otherwise $x_k = y_k = 0$, and we don't have a nontrivial linear combination). Thus, $\mathbf{a}_k$ is proportional to $\mathbf{a}_l$, $\mathbf{a}_k = z_k\mathbf{a}_l$, $z_k = -y_k/x_k$. Denoting $B = \mathbf{a}_l$, $C = (z_1, z_2, \ldots, z_{l-1}, 1, z_{l+1}, \ldots, z_m)$, we get $A = BC$.

5. We have $\mathrm{rk}(\beta \circ \alpha) \leqslant \mathrm{rk}(\beta)$ because clearly we have an inclusion of subspaces $\mathrm{Im}(\beta \circ \alpha) \subset \mathrm{Im}(\beta)$: every vector of the form $\beta(\alpha(u))$ with $u \in U$ is automatically a vector of the form $\beta(v)$ with $v \in V$. Also, we have $\mathrm{rk}(\beta \circ \alpha) \leqslant \mathrm{rk}(\alpha)$ because we can consider the map of vector spaces from $\mathrm{Im}(\alpha)$ to W induced by β; by the rank-nullity theorem, the rank of this map (equal to the rank of $\beta \circ \alpha$) does not exceed $\dim \mathrm{Im}(\alpha) = \mathrm{rk}(\alpha)$.