MA 1111: Linear Algebra I
Tutorial problems, October 4, 2018

1. (**a**) The easiest thing to do is to apply the algorithm from the lecture: take the matrix $(A \mid I)$ and bring it to the reduced row echelon form; for an invertible matrix A, the result is $(I \mid A^{-1})$. In this case, the inverse is $\begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$.

(**b**) If $A\mathbf{x} = \mathbf{b}$, then, multiplying by A^{-1}, we get $\mathbf{x} = A^{-1}\mathbf{b}$, so

$$\begin{cases} x_1 = 3a - 3b + c, \\ x_2 = -\frac{5}{2}a + 4b - \frac{3}{2}c, \\ x_3 = \frac{1}{2}a - b + \frac{1}{2}c. \end{cases}$$

2. Let us write down the corresponding one-row representatives of these permutations (for which we permute the columns to create the natural order in the top row): the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$ corresponds to $2, 1, 5, 4, 3$, the matrix $\begin{pmatrix} 1 & 4 & 2 & 3 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$ corresponds to $2, 5, 3, 1, 4$, and the matrix $\begin{pmatrix} 5 & 3 & 1 & 4 & 2 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$ corresponds to $2, 1, 5, 4, 3$. Therefore the first and the third matrix do represent the same permutation. The permutation $2, 1, 5, 4, 3$ is even (it has 4 inversions), and the permutation $2, 5, 3, 1, 4$ is odd (it has 3 inversions).

3. Clearly, we must have $i = 2$ (to have all the numbers present in the top row) and $\{j, k\} = \{3, 4\}$. For the choice $j = 3$, $k = 4$, the permutation is even (since there are 8 inversions in total in the two rows), so for the other choice the permutation is odd. *Answer:* $i = 2$, $j = 4$, $k = 3$.

4. (**a**) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$. Since $A^2 = I$, we have $b(a+d) = c(a+d) = 0$. If $a + d = 0$, we have $\operatorname{tr}(A) = 0$, and everything is proved. Otherwise, if $a + d \neq 0$, we have $b = c = 0$, so $a^2 = 1 = d^2$, and either $a = d = 1$ or $a = d = -1$ or $a = 1, d = -1$ or $a = -1, d = 1$. In the first case $A = I_2$, in the second case $A = -I_2$, in the remaining two cases $\operatorname{tr}(A) = a + d = 0$ (contradiction since we assumed $a + d \neq 0$). (**b**) For example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

5. Let (y_1, y_2) and (z_1, z_2) be two different solutions. Then $(y_1 - z_1, y_2 - z_2)$ is a solution to the system

$$\begin{cases} ax_1 + bx_2 = 0, \\ cx_1 + dx_2 = 0. \end{cases}$$

Indeed, $a(y_1 - z_1) + b(y_2 - z_2) = ay_1 + by_2 - az_1 - bz_2 = e - e = 0$, and the same for the second equation. This solution is different from $(0, 0)$ because the original solutions were different.