1111: Linear Algebra I
Selected exam questions from past years

November 29, 2018

1. (a) Compute the area of the parallelogram determined by the vectors $\mathbf{a} = (3, 5, -1)$ and $\mathbf{b} = (2, 0, 1)$.
   (b) Compute the angle between the diagonals of the above parallelogram.
2. Is the vector $\mathbf{n} = (-1, -3, 4)$ parallel to the intersection line of the plane α passing through the points $(0, 1, -1)$, $(5, 1, -3)$, and $(2, -3, 3)$ and the plane β passing through the points $(2, 1, -1)$, $(5, 1, -3)$, and $(1, -2, 3)$? Why?
3. Denote by A the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ and by b the vector $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$.
   (a) Show how to compute the matrix A^{-1} using elementary row operations, and use the matrix A^{-1} to solve the system $Ax = b$.
   (b) Show how to use the Cramer's rule to solve the system $Ax = b$.
4. Consider the system of linear equations

$$\begin{cases} 4x_1 - 2x_2 + 2x_3 = 5, \\ 2x_1 - x_2 + 3x_3 = 2, \\ 7x_1 - 2x_2 + 3x_3 = 6. \end{cases}$$

   (a) Use reduced row echelon forms to solve this system.
   (b) Compute the inverse matrix using the adjugate matrix formula, and use it to solve this system.
5. Denote by A the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

   and by b the vector $\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$. List all minors and all cofactors of A, and write down the expansion of $\det(A)$ along the second row and along the third column. Show how to use the Cramer's rule to solve the system $Ax = b$.
6. Using elementary row operations, compute the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}$. Find a polynomial $f(t)$ of degree at most 2 for which $f(1) = 1$, $f(3) = 0$, $f(4) = 11$.
7. In this question, A and B are $n \times n$-matrices. For each of the following statements, prove it if it is true, and give a counterexample if it is false.
   (a) $A^2 - B^2 = (A + B)(A - B)$ if and only if $AB = BA$.
   (b) If $A^2 = B^2$, then $A = B$ or $A = -B$.

8. (a) Which permutations are called even, and which — odd? Write down the corresponding definitions.
   (b) Does the product $a_{35}a_{21}a_{46}a_{17}a_{73}a_{54}a_{62}$ occur in the expansion of the 7×7-determinant? If yes, what is the coefficient of this product there? Answer the same questions for $a_{34}a_{15}a_{67}a_{26}a_{73}a_{51}a_{62}$.
   (c) Find i, j, and k for which the product $a_{51}a_{i6}a_{1j}a_{35}a_{44}a_{6k}$ occurs in the expansion of the 6×6-determinant with coefficient (-1).
9. (a) Outline the proof from class of the fact the determinant of a matrix does not change if we add to one of its rows a multiple of another row.
   (b) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & -3 \\ -3 & -4 & 1 & 2 \\ -4 & 3 & -2 & 1 \end{pmatrix}$, compute the matrix product of A and its transpose matrix, and explain how to use your result to calculate the determinant of A.
10. Prove that three points in the plane whose coordinates are (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively, all belong to the same line if and only if

$$\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} = 0.$$

11. Determine all values of x for which the matrix

$$\begin{pmatrix} 2-x & 1 & 0 \\ -1 & -x & 1 \\ 5 & 5 & 3-x \end{pmatrix}$$

is not invertible.
12. Let A and B be $n\times n$-matrices. Which of the following statements are true:
   (a) If AB is invertible, then either A is invertible or B is invertible.
   (b) If AB is invertible, then A is invertible and B is invertible.
   (c) If either A is invertible or B is invertible, then AB is invertible.
   (d) If A is invertible and B is invertible, then AB is invertible.
13. Compute the determinant of the $n\times n$-matrix $\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \dots & \vdots \\ \vdots & \dots & \dots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$ (all diagonal entries are equal to 0, all off-diagonal entries are equal to 1).
14. (a) Prove that for two square matrices A and B of the same size we always have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
   (b) How many *distinct* numbers can there be among the six traces

$$\mathrm{tr}(ABC), \mathrm{tr}(ACB), \mathrm{tr}(BCA), \mathrm{tr}(BAC), \mathrm{tr}(CBA), \mathrm{tr}(CAB)?$$

   for different choices of square matrices A, B, C of the same size? For each variant of the answer, give an example.
15. Assume that A is a 2×2-matrix with $A^2 - 3A + 2I_2 = 0$. What are possible values of $\mathrm{tr}(A)$?
16. Let us consider matrices B which commute with the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, that is $AB = BA$. How many rows and columns may such matrix have? Show that the set of all such matrices forms a vector space with respect to the usual matrix operations. Compute the dimension of that space.

17. Determine all values of x for which the three vectors

$$\begin{pmatrix} 2-x \\ -1 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -x \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 3-x \end{pmatrix}$$

are linearly dependent.

18. For the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 1 & 1 & 2 & 5 \\ -1 & 2 & 1 & 1 & 0 & 1 & 1 \end{pmatrix},$$

compute the dimension and find a basis of the solution space to the system of equations $Ax = 0$.

19. A 3×3-checkboard whose cells are filled in with 9 real numbers is called a magic square if all its row sums are pairwise equal, and equal to all of its column sums. Prove that the set of all magic squares forms a subspace of $\mathbb{R}^9$, compute the dimension of this space, and find a basis of this space.

20. (a) State the definition of a basis of a vector space. Show that the vectors $f_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ form a basis of $\mathbb{R}^2$.

    (b) Suppose that the matrix of a linear transformation of $\mathbb{R}^2$ relative to the basis f_1, f_2 from (a) is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Compute the matrix of the same linear transformation relative to the basis of standard unit vectors of $\mathbb{R}^2$.

21. Show that the map of the space P_2 of all polynomials in x of degree at most 2 to the same space that takes every polynomial $f(x)$ to $3x^2f''(x)+3f(x-1)$ is a linear transformation, and compute the matrix of that transformation relative to the basis $1, x+1, (x+1)^2$.

22. (a) State the definition of an eigenvalue and of an eigenvector of a linear transformation of a vector space V.

    (b) What are the eigenvalues of the linear transformation T of the four-dimensional space of 2×2-matrices which sends every matrix X to $AX - XA$, where $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$?

    (c) Does the linear transformation T from (b) have a basis of eigenvectors?